Spring 2021 Math 33A Exam 2

Instructions:

- You should complete the exam and submit by 8am on May 11th PDT. Please leave enough time to scan your work into a PDF and upload it to Gradescope! Exams will not be accepted after 8am.
- You may spend as much time as you like on this exam between now and the due date.
- If you do not have a way to print this exam you can copy each question onto a blank piece of paper. Please copy the question, and give yourself plenty of space to answer each question.
- This exam is open book. You can use any resources you find in our textbook, on our CCLE page or on the internet in general. However you should not have anyone's help to do the exam. So you are not allowed to ask your classmates or TAs about the questions or to post the exam questions on an online forum. Posting our exam questions online is the same asking someone to do the exam for you which is cheating.
- If you have a question about the phrasing of the one of the questions or about the mechanics of completing the exam, you can email ProfRosesMathExamQuestions@gmail.com.

By signing below, you certify that the following exam is entirely your own work and that you did not receive any help in completing the exam. You will not post the exam questions on public or class forums or discuss the questions with anyone else until after the exam has ended.

Eric Tang

(Signature)

Remember to explain your calculations for full credit! It's fine to use technology to check your answer, but please write out some intermediate steps in your row reductions so we can see your process!

OVER →

1. (8 pts) Consider the matrix

$$A = \begin{bmatrix} 2 & -3 & 1 \\ -2 & 3 & -1 \\ 1 & 6 & 8 \\ 0 & 1 & 1 \end{bmatrix}$$

(a) Find a basis for Im(A). Be sure to show your work.

$$\text{rref}(A) = \begin{bmatrix} 2 & -3 & 1 \\ 0 & 0 & 0 \\ 1 & 6 & 8 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 8 \\ 0 & 15 & 15 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 8 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

column 3 doesn't have a leading one, therefore it's redundant.

$$\text{Im}(A) = \text{span}\left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 6 \\ 1 \end{bmatrix} \right\}$$

A basis for Im(A) is $\begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ 6 \\ 1 \end{bmatrix}$

(b) Fill in the blanks below

The image of A is a plane (line/plane/other space) in $\mathbb{R}^4$ ($\mathbb{R}^n$ for what n)

OVER →

Question 1 continued....

(c) Use Gram Schmidt to find an orthonormal basis for the image of A. Be sure to show your work.

$$\vec{v_1} = \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix} \quad \vec{v_2} = \begin{bmatrix} -3 \\ 3 \\ 6 \\ 1 \end{bmatrix}$$

$$\vec{u_1} = \frac{\vec{v_1}}{\|\vec{v_1}\|} = \frac{1}{\sqrt{4+4+1}} \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \\ 0 \end{bmatrix}$$

$$\vec{v_2}^{\perp} = \begin{bmatrix} -3 \\ 3 \\ 6 \\ 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix} \cdot [-3\ 3\ 6\ 1] \cdot \frac{1}{3} \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 3 \\ 6 \\ 1 \end{bmatrix} + \frac{1}{9} \cdot 6 \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 6 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{4}{3} \\ -\frac{4}{3} \\ \frac{2}{3} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{5}{3} \\ \frac{5}{3} \\ \frac{20}{3} \\ 1 \end{bmatrix}$$

$$\vec{u_2} = \frac{\vec{v_2}^{\perp}}{\|\vec{v_2}^{\perp}\|} = \frac{1}{\sqrt{(\frac{5}{3})^2 \cdot 2 + (\frac{20}{3})^2 + 1}} \begin{bmatrix} -\frac{5}{3} \\ \frac{5}{3} \\ \frac{20}{3} \\ 1 \end{bmatrix} = \begin{bmatrix} -5/3\sqrt{51} \\ 5/3\sqrt{51} \\ 20/3\sqrt{51} \\ 1/\sqrt{51} \end{bmatrix}$$

(d) Find the vector $\bar{v}$ in Im(A) such that the distance between $\bar{v}$ and $(1, 1, 1, 1)$ is minimized.

$$\vec{V} = \mathrm{Proj}_{\mathrm{Im}(A)} (1,1,1,1)$$

$$= P \cdot (1,1,1,1)$$

$$P = QQ^T = \begin{bmatrix} 2/3 & -5/3\sqrt{51} \\ -2/3 & 5/3\sqrt{51} \\ 1/3 & 20/3\sqrt{51} \\ 0 & 1/\sqrt{51} \end{bmatrix} \begin{bmatrix} 2/3 & -2/3 & 1/3 & 0 \\ -5/3\sqrt{51} & 5/3\sqrt{51} & 20/3\sqrt{51} & 1/\sqrt{51} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{229}{459} & \frac{-229}{459} & \frac{2}{459} & \frac{-5}{153} \\ \frac{-229}{459} & \frac{229}{459} & \frac{-2}{459} & \frac{5}{153} \\ \frac{2}{459} & \frac{-2}{459} & \frac{451}{459} & \frac{20}{153} \\ \frac{-5}{153} & \frac{5}{153} & \frac{20}{153} & \frac{1}{51} \end{bmatrix}$$

$$P \cdot (1,1,1,1) = \begin{bmatrix} \frac{2}{459} - \frac{5}{153} \\ \frac{-2}{459} + \frac{5}{153} \\ \frac{451}{459} + \frac{20}{153} \\ \frac{20}{153} + \frac{1}{51} \end{bmatrix} = \begin{bmatrix} \frac{-13}{459} \\ \frac{13}{459} \\ \frac{511}{459} \\ \frac{23}{153} \end{bmatrix}$$

OVER →

2. (6 points) For each of the following, give an example with the described property and write a sentence or two explaining why your example satisfies the property. If no such example exists, give a reason why this is impossible.

(a) A 2×3 matrix such that kernel of A is Span$\{(1,2,3)\}$.

$$A = \begin{array}{c} \begin{matrix} x & y & z \end{matrix} \\ \begin{bmatrix} 3 & 0 & -1 \\ 0 & \frac{3}{2} & -1 \end{bmatrix} \end{array} \qquad \text{rref}(A) = \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \end{bmatrix} \qquad \begin{array}{c} \begin{matrix} x & y & z & \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \end{array}\right] \end{array}$$

$$z = t,\ x = \tfrac{1}{3}t,\ y = \tfrac{2}{3}t,\quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 1 \end{pmatrix} t \qquad \ker(A) = \text{span}\left\{\begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}$$

The kernel of the 2×3 matrix $\begin{bmatrix} 3 & 0 & -1 \\ 0 & \frac{3}{2} & -1 \end{bmatrix}$ spans $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(b) A matrix that is similar to the matrix A below, but not equal to it.

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$$

$$B = SAS^{-1} \qquad S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \cdot -1 \cdot \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

B is similar to A since it satisfies $B = SAS^{-1}$ and $B \neq A$

4 (c) A subspace of $\mathbb{R}^2$ containing the vectors $(1,0)$ and $(1,1)$, but not the vector $(2,1)$.

If a subspace in $\mathbb{R}^2$ contains two independent vectors, it means that it spans the entire $\mathbb{R}^2$, and any vector in $\mathbb{R}^2$ can be represented in a linear combination of these two vectors. (i.e. they form a basis)

$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Therefore, no such subspace exists

OVER →

3. (4 points)

Suppose that $\mathcal{B} = \{(1,4),(-4,1)\}$ is a basis for $\mathbb{R}^2$.

(a) Find the $\mathcal{B}$ coordinates for the vector $(-9,-2)$

$$\begin{bmatrix} -9 \\ -2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -4 & -9 \\ 4 & 1 & -2 \end{array}\right] = \left[\begin{array}{cc|c} 1 & -4 & -9 \\ 0 & 17 & 34 \end{array}\right] = \left[\begin{array}{cc|c} 1 & -4 & -9 \\ 0 & 1 & 2 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right] \qquad c_1 = -1,\ c_2 = 2$$

$$\begin{bmatrix} -9 \\ -2 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

(b) If $[\bar{v}]_{\mathcal{B}} = (2,3)$ is a vector in $\mathcal{B}$ coordinates, write the vector $\bar{v}$ using standard coordinates.

$$\vec{v} = 2 \cdot \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 3 \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 8 \end{bmatrix} + \begin{bmatrix} -12 \\ 3 \end{bmatrix} = \begin{bmatrix} -10 \\ 11 \end{bmatrix}$$

OVER →

4. (4pts) Determine if each of the following statements is True or False and give a justification.

(a) If $A\bar{x} = \bar{0}$ has infinitely many solutions then the columns of A must be dependent. (note the size of A is unknown, it may be square, but it may not be square)

True. If $A\vec{x} = \vec{0}$ has infinitely many solutions, rank(A) < number of columns of A.

If $A\vec{x} = 0$ has an unique solution, $\ker(A) = \vec{0}$, rank(A) = number of columns of A, and the columns are independent. Since rank(A) < number of columns of A, the columns of A must be dependent.

(b) If A and B are both orthogonal transformations then $A^{-1}B^T$ is also orthogonal.

If A and B are orthogonal, $A^T = A^{-1}$, $B^T = B^{-1}$, and

$A^{-1}B^T = A^{-1}B^{-1} = (BA)^{-1}$

If A and B are orthogonal, then BA is also orthogonal

If a matrix M is orthogonal, M^{-1} is orthogonal

Since BA is orthogonal, $(BA)^{-1} = A^{-1}B^T$ is orthogonal.

The statement is true

OVER →

5. (6 points) Suppose that $\bar{v}_1$ and $\bar{v}_2$ are linearly independent vectors in $\mathbb{R}^3$ and $\bar{v}_3$ is a linear combination of $\bar{v}_1$ and $\bar{v}_2$. For each of the following either determine the dimension of the given subspace and justify this calculation, or explain why there is not enough information to determine this.

(a) $Span\{v_1, v_2, v_3\}$

Since $\vec{v_3}$ is a linear combination of $\vec{v_1}, \vec{v_2}$, it's redundant.

$\text{Span}\{\vec{v_1}, \vec{v_2}, \vec{v_3}\} = \text{Span}\{\vec{v_1}, \vec{v_2}\}$

$\dim(\text{Span}\{\vec{v_1}, \vec{v_2}\}) = 2$ because there are two linearly independent vectors

(b) $Span\{v_1, v_3\}$

$\dim(\text{Span}\{\vec{v_1}, \vec{v_3}\}) = 1$ or 2. If $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$,

$\vec{v_3} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = 3\vec{v_1} + 0\vec{v_2}$, then $\text{span}\{\vec{v_1}, \vec{v_3}\} = \text{span}\{\vec{v_1}\}$

because $\vec{v_3}$ is a multiple of $\vec{v_1}$, and the dimension of the span $= 1$. If $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v_3} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \vec{v_1} + \vec{v_2}$,

then $\dim(\text{span}\{\vec{v_1}, \vec{v_3}\}) = 2$ because $\vec{v_1}, \vec{v_3}$ are independent.

(c) $Span\{v_1, v_1 + v_2\}$

$\dim(\text{span}\{\vec{v_1}, \vec{v_1} + \vec{v_2}\}) = 2$. $\vec{v_1} + \vec{v_2}$ is never a nonzero multiple of $\vec{v_1}$ given that $\vec{v_1}, \vec{v_2}$ are linearly independent, so $\vec{v_1}$ and $\vec{v_1} + \vec{v_2}$ are independent