1. (6 pts) Consider a quadratic polynomial (a function of the form $f(x) = ax^2 + bx + c$) which intersects the points (1,-2), (-1,-6), (-2,-11) and (2,-3). Does such a polynomial exist? If so, find all possible values for $a, b$ and $c$. Be sure to show the steps of your calculation including any row reductions.

For an intersection, there exists a polynomial where $y = ax^2+bx+c$ for the point $(x,y)$ (the polynomial includes that point), so for intersecting the 4 points there must be a common polynomial (with common $a, b, c$) containing the points.

Point (1,-2)
$-2 = a(1)^2+b(1)+c$
$-2 = a+b+c$

Point (-1,-6)
$-6 = a(-1)^2+b(-1)+c$
$-6 = a-b+c$

Point (-2,-11)
$-11 = a(-2)^2+b(-2)+c$
$-11 = 4a-2b+c$

Point (2,-3)
$-3 = a(2)^2+b(2)+c$
$-3 = 4a+2b+c$

System of linear equations:

$a+b+c = -2$
$a-b+c = -6$
$4a-2b+c = -11$
$4a+2b+c = -3$

Using elimination to find the rref

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 1 & -1 & 1 & -6 \\ 4 & -2 & 1 & -11 \\ 4 & 2 & 1 & -3 \end{array}\right] \begin{array}{c} \\ \xrightarrow{-I} \\ \xrightarrow{-4I} \\ \xrightarrow{-4I} \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -2 & 0 & -4 \\ 0 & -6 & -3 & -3 \\ 0 & -2 & -3 & 5 \end{array}\right] \begin{array}{c} \\ \xrightarrow{\div -2} \\ \xrightarrow{\div -3} \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & -2 & -3 & 5 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & -2 & -3 & 5 \end{array}\right] \begin{array}{c} \xrightarrow{-II} \\ \\ \xrightarrow{-2II} \\ \xrightarrow{+2II} \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & -3 & 9 \end{array}\right] \begin{array}{c} \\ \\ \\ \xrightarrow{\div -3} \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & -3 \end{array}\right] \begin{array}{c} \xrightarrow{-III} \\ \\ \\ \xrightarrow{-III} \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Final rref (augmented matrix): $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$a = -1$
$b = 2$
$c = -3$

One unique solution (rank $= 3$)

Yes, the polynomial exists because there is a solution to the system of linear equations representing the possible intersection of 4 planes or a polynomial with each point contained.

Polynomial: $f(x) = -x^2+2x-3$

$a = -1$
$b = 2$
$c = -3$

OVER →

2. (5 pts) Consider the following linear transformations:

$T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(x_1, x_2, x_3) = (x_1 + 2x_2 + 3x_3, -x_1 - x_2 - x_3, x_1 + x_2 + x_3)$

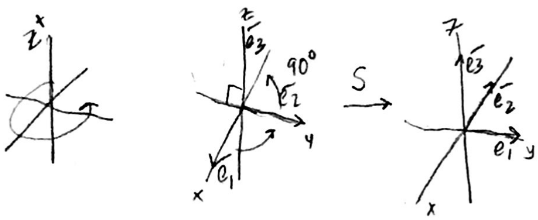
$S : \mathbb{R}^3 \to \mathbb{R}^3$ which rotates by $90°$ about the z-axis (counterclockwise as viewed from the positive z-axis).

(a) Find a matrix $A$ such that $T(\bar{x}) = A\bar{x}$ and a matrix $B$ such that $S(\bar{x}) = B\bar{x}$

$$A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ -x_1 - x_2 - x_3 \\ x_1 + x_2 + x_3 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

dot product

$x_1 T(\bar{e}_1) \quad x_2 T(\bar{e}_2) \quad x_3 T(\bar{e}_3)$ $\qquad$ $T(\bar{e}_1) \quad T(\bar{e}_2) \quad T(\bar{e}_3)$




$$B = \begin{bmatrix} T(\bar{e}_1) & T(\bar{e}_2) & T(\bar{e}_3) \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Compute a matrix $C$ such that composition $S \circ T(\bar{x}) = C\bar{x}$.

$T(\bar{x}) = A\bar{x}$

$S(T(\bar{x})) = B(A\bar{x}) = BA\bar{x}$ $\qquad$ $BA = C$

$$BA = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

(c) Is the matrix C invertible? Why or Why not?

Find rref(C)

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[-I]{-I} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-II} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix is <u>not</u> invertible since rref(C) $\neq I_n$ or $I_3$ and rank(C) $= 2 \neq 3$. The characterization of an invertible square matrix is not fufilled.

**OVER →**

3. (5) Consider the following matrix.

$$\begin{bmatrix} a & 2 & 0 & 0 \\ 0 & b & c & 1 \\ d & 0 & 0 & 0 \end{bmatrix}$$

If we know this matrix is a rref, can you determine the values of $a, b, c$ and $d$? For each of these variables either

i) determine what value that variable must take and justify this

or

ii) give two distinct rrefs with different values for the given variable.

For a: i) $\begin{bmatrix} a & 2 & 0 & 0 \\ 0 & b & c & 1 \\ d & 0 & 0 & 0 \end{bmatrix}$ No leading 1 in the row and first nonzero entry must be 1, so a must be 1.

$a=1$

For d: i) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & b & c & 1 \\ d & 0 & 0 & 0 \end{bmatrix}$ Since $a=1$, there can only be 0s for the other entries in the column so d must be 0.

$d=0$

For b: i) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & b & c & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ b cannot be a leading 1 since there is a nonzero entry in the column ('2') and b cannot be a nonzero entry not 1 since b would be the first nonzero entry in the row, so b must be equal to 0.

$b=0$

For c: ii) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & c & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Since the other entries in the column are 0s and c can be the first non zero entry, c can be a leading 1.

$c=1$

$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & c & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Since the column right to c has a possible leading 1 if $c=0$ (could be first nonzero entry and has 0s in column), c could be a 0.

$c=0$

$c=1$

$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$c=0$

$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

**OVER →**

4. (12 pts) Give examples with each of the following properties, and explain briefly (1-2 sentences), why the example you chose has the desired property. If no such example exists, use one of our theorems from class to explain why this can't happen.

(a) An example of a 4x3 matrix $A$ where $A\bar{x} = \bar{0}$ has a unique solution.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Coefficient Matrix of $A\bar{x} = \bar{0}$ has a unique solution since rank$(A) = 3$ or number of columns $m$ and last row is a redundant equation in the form $[0\,0\,0 : 0]$ (so not no solution).

Only solution is $\bar{x} = \bar{0}$.

(b) An example of a system of linear equations that does not have infinitely many solutions, but where the rank of the coefficient matrix is less than the number of variables.

3x3 coefficient matrix with rank $= 2$

$$\left[\begin{array}{ccc|c} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$0 \neq 1$ so no solutions.

EX: System of linear equations

$x_1 + 0 + x_3 = 1$

$0 + x_2 + x_3 = -1$

$0 + x_2 + x_3 = 0$

parallel lines

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{-\text{II}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad 0 \neq 1$$

The rank of the coefficient matrix is less than the number of variables but since the last equation is parallel and not equal to a previous equation there are no solutions.

(c) An example of matrix $B$ where $B^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

No example

No example since $B^{-1}$ or $B$ is not invertible

$BB^{-1} = I_n$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Theorem: $B$ such that $BB^{-1} = I_2$ then $B$ and $B^{-1}$ are invertible matrixes. There is no $B$ s.t that holds,

$a+b=1 \quad a+b=0$

$c+d=0 \quad c+d=1$

no solution

Theorem implies that $B^{-1}(B^{-1})^{-1} = I_n$ (or $B^{-1}B = I_n$). To find the inverse you can find rref$[B|I_n]$ that yields $[I_n|B^{-1}]$, and in this case it was not in that form.

$B^{-1}(B^{-1})^{-1} = I_n$

$$\left[\begin{array}{cc|c} 1 & 1 & y_1 \\ 1 & 1 & y_2 \end{array}\right] \xrightarrow{-\text{I}} \left[\begin{array}{cc|c} 1 & 1 & y_1 \\ 0 & 0 & -y_1 + y_2 \end{array}\right]$$

$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \neq I_n$

Similarly, $\det(B^{-1}) = 1 - 1 = 0$ for 2x2 matrices so there is no $(B^{-1})^{-1}$ or $B$.

**OVER →**

(d) An example of a vector $\bar{v}$ in $\mathbb{R}^3$, which is not a linear combination of the vectors $(1, 1, 0)$ and $(0, 0, 1)$

$$\boxed{\bar{v} = \begin{pmatrix}1\\2\\3\end{pmatrix}}$$

Solve $A\bar{x} = \bar{b}$ $\quad \bar{v} = a\bar{x} + b\bar{y}$
to see if possible linear combination

$$\bar{x} = \begin{bmatrix}1\\1\\0\end{bmatrix} = 1\begin{pmatrix}1\\0\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\bar{y} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\left[\begin{array}{cc|c}1&0&1\\1&0&2\\0&1&3\end{array}\right] \xrightarrow{-I} \left[\begin{array}{cc|c}1&0&1\\0&0&1\\0&1&3\end{array}\right] \text{swap} \left[\begin{array}{cc|c}1&0&1\\0&1&3\\0&0&1\end{array}\right]$$

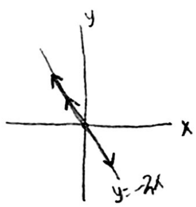
$0 \neq 1$
no solution

$$\bar{v} = a\bar{x} + b\bar{y}$$
$$\bar{v} = a\begin{pmatrix}1\\1\\0\end{pmatrix} + b\begin{pmatrix}0\\0\\1\end{pmatrix}$$
$$\bar{v} = a\left(\begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix}\right) + b\begin{pmatrix}0\\0\\1\end{pmatrix}$$
$$\bar{v} = a\begin{pmatrix}1\\0\\0\end{pmatrix} + a\begin{pmatrix}0\\1\\0\end{pmatrix} + b\begin{pmatrix}0\\0\\1\end{pmatrix}$$

can only be in form $\bar{v} = \begin{pmatrix}a\\a\\b\end{pmatrix}$

vector $\begin{pmatrix}1\\2\\3\end{pmatrix}$ is not a linear combination since there is no solution to $\begin{pmatrix}1\\2\\3\end{pmatrix} = a\begin{pmatrix}1\\1\\0\end{pmatrix} + b\begin{pmatrix}0\\0\\1\end{pmatrix}$ and since the only possible combinations are in form $\begin{pmatrix}a\\a\\b\end{pmatrix}$.

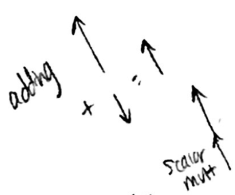
(e) An example of two vectors $\bar{v}_1$ and $\bar{v}_2$ such that the set of all linear combinations of $\bar{v}_1$ and $\bar{v}_2$ is equal to the vectors on the line $y = -2x$.




For any two vectors $\bar{v}_1$ and $\bar{v}_2$ on the line, $\bar{v}_1 + \bar{v}_2$ is on the line and $k\bar{v}_1$ or $k\bar{v}_2$ is on the line, so (k is scalar)

$a\bar{v}_1 + b\bar{v}_2$ is on the line. (a, b are scalars).

$A\bar{x} = \bar{0}$ where the columns of A are two parallel vectors on the line yields a solution set that is the line.




$x = 1$, $y = -2$ $\qquad$ $y = 1$, $x = -\frac{1}{2}$

$$\boxed{\bar{v}_1 = \begin{pmatrix}1\\-2\end{pmatrix} \quad \bar{v}_2 = \begin{pmatrix}-\frac{1}{2}\\1\end{pmatrix}}$$

$y = -2x$ so $-2x + y = 0$ or $x - \frac{1}{2}y = 0$
$A\bar{x} = \bar{0}$

$$\left[\begin{array}{cc|c}1&-\frac{1}{2}&0\\-2&1&0\end{array}\right] \xrightarrow{+2I} \left[\begin{array}{cc|c}1&-\frac{1}{2}&0\\0&0&0\end{array}\right]$$

$x - \frac{1}{2}y = 0$
Solutions: $y = -2x$

(f) An example of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(e_1) = (1, 2, 3)$, $T(e_2) = (-2, -4, -6)$ and $T(1, 1) = (0, 0, 0)$

No example

No example exists.

For a linear transformation $\mathbb{R}^m$ to $\mathbb{R}^n$ there exists an $n \times m$ matrix A s.t $T(\bar{x}) = A\bar{x}$ and $A = [T(\bar{e}_1)\ T(\bar{e}_2) \ldots T(\bar{e}_m)]$. $A = [T(\bar{e}_1)\ T(\bar{e}_2)] = \begin{bmatrix}1&-2\\2&-4\\3&-6\end{bmatrix}$. However $A\bar{x} = \bar{0}$ does not include $\bar{x} = \begin{pmatrix}1\\1\end{pmatrix}$ $(T(1,1) \neq \bar{0})$ so no transformation exists.

Solution to $A\bar{x} = \bar{0}$:

$$\begin{bmatrix}1&-2\\2&-4\\3&-6\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}1&-2\\2&-4\\3&-6\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-1\\-2\\-3\end{bmatrix} \neq \bar{0}$$

$$\left[\begin{array}{cc|c}1&-2&0\\2&-4&0\\3&-6&0\end{array}\right] \begin{array}{c}\xrightarrow{-2I}\\ \xrightarrow{-3I}\end{array} \left[\begin{array}{cc|c}1&-2&0\\0&0&0\\0&0&0\end{array}\right]$$

free variable
$\infty$ solutions

$x - 2y = 0$
$1 - 2 = -1 \neq 0$