Spring 2021 Math 33A Exam 1

Instructions:

- You should complete the exam and submit by 8am on April 20th PDT. Please leave enough time to scan your work into a PDF and upload it to Gradescope! Exams will not be accepted after 8am.
- You may spend as much time as you like on this exam between now and the due date.
- If you do not have a way to print this exam you can copy each question onto a blank piece of paper. Please copy the question, and give yourself plenty of space to answer each question.
- This exam is open book. You can use any resources you find in our textbook, on our CCLE page or on the internet in general. However you should not have anyone's help to do the exam. So you are not allowed to ask your classmates or TAs about the questions or to post the exam questions on an online forum. Posting our exam questions online is the same asking someone to do the exam for you which is cheating.
- If you have a question about the phrasing of the one of the questions or about the mechanics of completing the exam, you can email ProfRosesMathExamQuestions@gmail.com.

By signing below, you certify that the following exam is entirely your own work and that you did not receive any help in completing the exam. You will not post the exam questions on public or class forums or discuss the questions with anyone else until after the exam has ended.

Marisa Duran
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(Signature)

Remember to explain your calculations for full credit! It's fine to use technology to check your answer, but please write out some intermediate steps in your row reductions so we can see your process!

OVER →

1. (6 pts) Consider a quadratic polynomial (a function of the form $f(x) = ax^2 + bx + c$) which intersects the points (1,-2), (-1,-6), (-2,-11) and (2,-3). Does such a polynomial exist? If so, find all possible values for a, b and c. Be sure to show the steps of your calculation including any row reductions.

Plugging in the points to $f(x)$:

$$a(1^2) + b(1) + c = -2 \rightarrow a + b + c = -2$$
$$a(-1)^2 + b(-1) + c = -6 \rightarrow a - b + c = -6$$
$$a(-2)^2 + b(-2) + c = -11 \rightarrow 4a - 2b + c = -11$$
$$a(2)^2 + b(2) + c = -3 \rightarrow 4a + 2b + c = -3$$

Putting the system of equations into an augmented matrix and finding the rref:

$$\text{rref}\begin{array}{c} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \\ \mathrm{IV} \end{array}\left[\begin{array}{ccc:c} 1 & 1 & 1 & -2 \\ 1 & -1 & 1 & -6 \\ 4 & -2 & 1 & -11 \\ 4 & 2 & 1 & -3 \end{array}\right] \begin{array}{l} \\ \mathrm{II} = \mathrm{II} - \mathrm{I} \\ \mathrm{III} = \mathrm{III} - 4\mathrm{I} \\ \mathrm{IV} = \mathrm{IV} - 4\mathrm{I} \end{array} \rightarrow \begin{bmatrix} 1 & 1 & 1 & -2 \\ 0 & -2 & 0 & -4 \\ 0 & -6 & -3 & -3 \\ 0 & -2 & -3 & 5 \end{bmatrix} \begin{array}{l} \\ \mathrm{II} = \mathrm{II}/-2 \\ \\ \\ \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & -6 & -3 & -3 \\ 0 & -2 & -3 & 5 \end{bmatrix} \begin{array}{l} \mathrm{I} = \mathrm{I} - \mathrm{II} \\ \\ \mathrm{III} = \mathrm{III} + 6\mathrm{II} \\ \mathrm{IV} = \mathrm{IV} + 2\mathrm{II} \end{array} \rightarrow \begin{bmatrix} 1 & 0 & 1 & -4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -3 & 9 \\ 0 & 0 & -3 & 9 \end{bmatrix} \begin{array}{l} \\ \\ \mathrm{III} = \mathrm{III}/-3 \\ \mathrm{IV} = \mathrm{IV}/-3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 1 & -4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & -3 \end{bmatrix} \begin{array}{l} \mathrm{I} = \mathrm{I} - \mathrm{III} \\ \\ \\ \mathrm{IV} = \mathrm{IV} - \mathrm{III} \end{array} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

from the rref, we find that

$a = -1$
$b = 2$
$c = -3$

OVER →

2. (5 pts) Consider the following linear transformations:

$T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(x_1, x_2, x_3) = (x_1 + 2x_2 + 3x_3, -x_1 - x_2 - x_3, x_1 + x_2 + x_3)$

$S : \mathbb{R}^3 \to \mathbb{R}^3$ which rotates by 90° about the z-axis (counterclockwise as viewed from the positive z-axis).

(a) Find a matrix A such that $T(\bar{x}) = A\bar{x}$ and a matrix B such that $S(\bar{x}) = B\bar{x}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

(from the coefficients of the equations)

$$B = \begin{bmatrix} \cos 90° & -\sin 90° & 0 \\ \sin 90° & \cos 90° & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(from the rotation in 2D)

(b) Compute a matrix C such that composition $S \circ T(\bar{x}) = C\bar{x}$.

$C\vec{x} = S \circ T(\vec{x}) = S(T(\vec{x})) = B(A\vec{x}) = (BA)\vec{x}$

$$C = BA = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0\cdot1 + (-1)\cdot(-1) + 0\cdot1 & 0\cdot2 + (-1\cdot-1) + 0\cdot1 & 0\cdot3 + (-1)\cdot(-1) + 0\cdot1 \\ 1\cdot1 + 0\cdot-1 + 0\cdot1 & 1\cdot2 + 0\cdot-1 + 0\cdot1 & 1\cdot3 + 0\cdot-1 + 0\cdot1 \\ 0\cdot1 + 0\cdot-1 + 1\cdot1 & 0\cdot2 + 0\cdot-1 + 1\cdot1 & 0\cdot3 + 0\cdot-1 + 1\cdot1 \end{bmatrix}$$

$$= \begin{bmatrix} 0+1+0 & 0+1+0 & 0+1+0 \\ 1+0+0 & 2+0+0 & 3+0+0 \\ 0+0+1 & 0+0+1 & 0+0+1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

(c) Is the matrix C invertible? Why or Why not?

Finding the rref of C to determine invertibility:

$$\text{rref}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{matrix} \text{II} = \text{II} - \text{I} \\ \text{III} = \text{III} - \text{I} \end{matrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{I} = \text{I} - \text{II}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

C is not invertible because its rref is not equivalent to the Identity matrix $\Bigg(\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I_3}\Bigg)$

OVER →

3. (5) Consider the following matrix.

$$\begin{bmatrix} a & 2 & 0 & 0 \\ 0 & b & c & 1 \\ d & 0 & 0 & 0 \end{bmatrix}$$

If we know this matrix is a rref, can you determine the values of a, b, c and d? For each of these variables either

i) determine what value that variable must take and justify this

or

ii) give two distinct rrefs with different values for the given variable.

if this matrix is a rref, a must be 1.

this is because the first row needs to have a leading one before the 2. then, because a is a leading 1, the rest of the first column must be 0, meaning that d must be 0.

b also must be 0, because as there is no leading 1 earlier in the row, it can only be 0 or a leading 1. however, b cannot be a leading 1 because the rest of the column would need to be zero, but there is a 2 in the first row. So b must be 0.

for c, the value could be zero or one.

the two potential rrefs are

$$\begin{bmatrix} a & 2 & 0 & 0 \\ 0 & b & 0 & 1 \\ d & 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} a & 2 & 0 & 0 \\ 0 & b & 1 & 1 \\ d & 0 & 0 & 0 \end{bmatrix}$$

where $a = 1$, $b = 0$, $d = 0$

OVER →

4. (12 pts) Give examples with each of the following properties, and explain briefly (1-2 sentences), why the example you chose has the desired property. If no such example exists, use one of our theorems from class to explain why this can't happen.

(a) An example of a 4x3 matrix A where $A\bar{x} = \bar{0}$ has a unique solution.

An example is $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 2 & 0 \end{bmatrix}$ This will have a unique solution. This can be proven by finding the rref of the augmented matrix A_m

$\text{rref}(A_m) = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 \end{array}\right] \xrightarrow{IV = IV - 2I} \left[\begin{array}{cccc} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & -4 & 0 \end{array}\right] \xrightarrow{IV = IV - 2II} \left[\begin{array}{cccc} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right] \xrightarrow[IV = IV + 4III]{I = I - 2III} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

the final rref shows that $x_1 = 0$, $x_2 = 0$, $x_3 = 0$.

there only 3 variables, so there are no free variables and $(0,0,0)$ is a unique solution

(b) An example of a system of linear equations that does not have infinitely many solutions, but where the rank of the coefficient matrix is less than the number of variables.

an example would be a system with equations

$x - y = 2$
$x - y = 3$

the coefficient matrix's rank would be 1, but there are 2 variables. This would mean there are no solutions, as proven by its rref:

$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 1 & -1 & 3 \end{array}\right] \xrightarrow{II = II - I} \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 0 & 1 \end{array}\right]$

so the coefficient matrix (the left two columns) has a rank of 1. the bottom row of the augmented matrix shows that there is no solution, as $0 \neq 1$

(c) An example of matrix B where $B^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

No such example exists.

say $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. for 2x2 matrices, their inverse is $\frac{1}{\det(B)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

so $B^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ meaning $a, d = 1$. and $b, c = -1$

$B^{-1} = \frac{1}{1 \cdot 1 - (-1)(-1)} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{1-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{0} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

the determinant of B cannot be zero, so it is impossible for B^{-1} to be an inverse matrix

OVER →

(d) An example of a vector $\bar{v}$ in $\mathbb{R}^3$, which is not a linear combination of the vectors $(1,1,0)$ and $(0,0,1)$

the vector $(1,0,1)$ is not a linear combination of these two vectors. There is no way for the vector $(1,1,0)$ to have different values in the first and second spots, and the vector $(0,0,1)$ will never have a nonzero number for the first two slots.

(e) An example of two vectors $\bar{v}_1$ and $\bar{v}_2$ such that the set of all linear combinations of $\bar{v}_1$ and $\bar{v}_2$ is equal to the vectors on the line $y = -2x$.

an example would be $\vec{v}_1 = (0,0)$ and $\vec{v}_2 = (1,-2)$. If this is the case, then any linear combination of $\vec{v}_1$ and $\vec{v}_2$ will lie on the line $y=-2x$, because $\vec{v}_2$ will always lie on this line and $\vec{v}_1$ will always be 0.

$a\vec{v}_1 + b\vec{v}_2 = \vec{0} + b(1,-2) = b(1,-2)$

Thm 1.3.10

(f) An example of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(e_1) = (1,2,3)$, $T(e_2) = (-2,-4,-6)$ and $T(1,1) = (0,0,0)$

there is no such example. there is a theorem that states that for T to be a linear transformation $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$. Taking $\vec{e}_1$ to be $\vec{v}$ and $\vec{e}_2$ to be $\vec{w}$, $T(\vec{e}_1 + \vec{e}_2) = T(\vec{e}_1) + T(\vec{e}_2)$ so $T(1,1) = T(\vec{e}_1) + T(\vec{e}_2)$ so $(0,0,0)$ must equal $(1,2,3) + (-2,-4,-6)$ for T to be a linear transformation. $(0,0,0) \neq (-1,-2,-3)$ so T is not a linear transformation, meaning there is no example.