21S-MATH33A-1 Final exam

MATTHEW NIEVA

TOTAL POINTS

50 / 50

QUESTION 1

1 Question 1 7 / 7

part a- out of 3 points

✓ **+ 3 pts All correct. including use of the fact that $$v\cdot w=0$$ to get $$a=\pm 3$$ and some argument about invertibility of A to conclude $$a=-3$$.**

+ **1.5 pts** Got $$a=\pm 3$$ but concluded that both options are valid.

+ **1 pts** Got $$a= 3$$ and thus did not find the right answer

+ **0 pts** Got wrong value for a/ no answer

+ **0.5 pts** Understood that $$v\cdot w=0$$

part b - out of 2 points

✓ **+ 2 pts Correct computation of $$A^{-1}$$ with steps showing row reduction**

+ **0 pts** Computed something else- you were asked for the matrix $$A^{-1}$$

+ **0 pts** Concluded that there is no $$A^{-1}$$

+ **2 pts** Got wrong value for a, but computed $$A^{-1}$$ correctly based on previous work.

+ **1.5 pts** Showed some steps to compute $$A^{-1}$$

part c out of 2 points

✓ **+ 2 pts Correct answer (48), with explanation**

+ **1 pts** Found det (A) instead

+ **0 pts** Incorrect (note $$det(2A)\neq 2det(A)$$)

+ **1 pts** Got wrong value for a or Det(A), but computed determinant correctly based on this value.

QUESTION 2

2 Question 2 6 / 6

✓ **+ 6 pts All correct.**

+ **1 pts** Correct computation of singular values

+ **1 pts** Correct computation of $$\Sigma$$ (based on sigular values from part a) Must be a 3x2 matix.

+ **2 pts** Correct computation of V (don't forget to normalize)

+ **2 pts** Correct computation of U(again this should be an orthogonal matrix)

+ **1 pts** correctly computed some columns or aspects of U, but is missing some columns or did not make the columns O.N. Or some columns are incorrect

+ **0 pts** See comments

+ **1 pts** Parts of V correct

+ **1 pts** Parts of Sigma correct

+ **0 pts** No credit

QUESTION 3

3 Question 3 5 / 5

✓ **+ 5 pts All correct**

+ **2 pts** Part a correct (Yes they are independent with justification)

+ **1 pts** Part a partially correct. (Yes they are independent but no justification or incorrect justification)

+ **2 pts** Part b correct (no they are no orthogonal with justification. Justification can be that A is not an orthogonal matrix, or give a counterexample)

+ **1 pts** Part b is partially correct (justification is not sufficient but the answer is correct)

+ **1 pts** Correct part c- with justification (i.e Area =4*det(A)=24

+ **0.75 pts** arithmetic error in part c, but otherwise correct.

+ **0 pts** All parts are wrong.

QUESTION 4

4 Question 4 4 / 4

✓ + **1 pts** Correct example

✓ + **1 pts** Justification that example is invertible

✓ + **1 pts** Justification that this example has at least one eigenvalue

✓ + **1 pts** Justification that this matrix is not diagoanlizable.

+ **0 pts** No answer

QUESTION 5

5 Question 5 **8 / 8**

✓ + **2 pts** Part a) not invertible with justification

✓ + **2 pts** part b) invertible with justification

✓ + **2 pts** part c) invertible with justification

✓ + **2 pts** part d) not invertible with justification.

+ **0 pts** incorrect justification for one or more parts.

QUESTION 6

6 Question 6 **9 / 9**

✓ - **0 pts** correct

- **1 pts** part a one eigenvalue wrong

- **1 pts** part b missing vector or too many vectors or one vector wrong

- **1 pts** part c needs more explanation

- **3 pts** part b vectors wrong

- **3 pts** part c wrong

- **2 pts** part b 2 vectors wrong

- **3 pts** part a wrong

- **2 pts** part a two eigenvalue wrong

- **9 pts** not found

QUESTION 7

7 Question 7 **8 / 8**

✓ + **8 pts** All correct

+ **3 pts** Part a Correct

+ **1 pts** part b correct

+ **2 pts** part c correct

+ **2 pts** Part d correct.

+ **1.5 pts** part d) 1:2 instead of 1:2 thousand

+ **0 pts** No credit

QUESTION 8

8 Signature **3 / 3**

✓ + **3 pts** Signature present.

1. (7 pts) Suppose that the vector $\bar{v} = (a, -2)$ and the vector $\bar{w} = (2a, 9)$ are orthogonal vectors.

Also suppose that the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & a & 1 \\ 0 & 6 & a-1 \end{bmatrix}$$

is invertible.

(a) Use this information to solve for a.

$\vec{v} \cdot \vec{w} = 0$ if orthogonal

$2a^2 - 18 = 0$

$2a^2 = 18$

$a^2 = 9$

$a = \pm 3$

Test A when A=3, when A=-3, see if det A ≠ 0 to be invertible.

$A_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 6 & 2 \end{bmatrix}$

$\det(A_3) = (1)(6-6) = 0$

$\det(A_3) = 0 \therefore A_3$ is not invertible

$A_{-3} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 6 & -4 \end{bmatrix}$

$\det(A_{-3}) = (1)(12-6) = 6$

$\det(A_{-3}) = 6 \quad 6 \neq 0 \therefore A_{-3}$ is invertible.

$\boxed{a = -3}$

(b) Use your answer from part a) to compute the matrix A^{-1}. (Be sure to show your work)

$A = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 & 1 & 0 \\ 0 & 6 & -4 & 0 & 0 & 1 \end{array}\right] \begin{matrix} 3I + II \\ \\ 2II + III \end{matrix}$

$= \left[\begin{array}{ccc|ccc} 3 & 0 & 4 & 3 & 1 & 0 \\ 0 & -3 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 & 2 & 1 \end{array}\right] \begin{matrix} I + 2III \\ 2II + III \\ \\ \end{matrix}$

$\rightarrow = \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 3 & 5 & 2 \\ 0 & -6 & 0 & 0 & 4 & 1 \\ 0 & 0 & -2 & 0 & 2 & 1 \end{array}\right] \begin{matrix} I/3 \\ II/-6 \\ III/-2 \end{matrix}$

$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 5/3 & 2/3 \\ 0 & 1 & 0 & 0 & -2/3 & -1/6 \\ 0 & 0 & 1 & 0 & -1 & -1/2 \end{array}\right]$

$\boxed{A^{-1} = \begin{bmatrix} 1 & 5/3 & 2/3 \\ 0 & -2/3 & -1/6 \\ 0 & -1 & -1/2 \end{bmatrix}}$

(c) Use your answer from part a) to compute the determinant of the matrix 2A.

If $A = \begin{bmatrix} \leftarrow I \rightarrow \\ \leftarrow II \rightarrow \\ \leftarrow III \rightarrow \end{bmatrix}$

$2A = \begin{bmatrix} \leftarrow 2I \rightarrow \\ \leftarrow 2II \rightarrow \\ \leftarrow 2III \rightarrow \end{bmatrix}$

$\det(A) = 6$

Row operations performed on A to get 2A: 2I, 2II, 2III

Track row operations from A to 2A

$\therefore \det(2A) = 2 \cdot 2 \cdot 2 \cdot \det(A)$

$= 8\det(A) = 48$

$\boxed{\det(2A) = 48}$

OVER →

1 Question 1 **7 / 7**

part a- out of 3 points

✓ **+ 3 pts** **All correct. including use of the fact that $$v\cdot w=0$$ to get $$a=\pm 3$$ and some argument about invertibility of A to conclude $$a=-3$$.**

- **+ 1.5 pts** Got $$a=\pm 3$$ but concluded that both options are valid.
- **+ 1 pts** Got $$a= 3$$ and thus did not find the right answer
- **+ 0 pts** Got wrong value for a/ no answer
- **+ 0.5 pts** Understood that $$v\cdot w=0$$

part b - out of 2 points

✓ **+ 2 pts** **Correct computation of $$A^{-1}$$ with steps showing row reduction**

- **+ 0 pts** Computed something else- you were asked for the matrix $$A^{-1}$$
- **+ 0 pts** Concluded that there is no $$A^{-1}$$
- **+ 2 pts** Got wrong value for a, but computed $$A^{-1}$$ correctly based on previous work.
- **+ 1.5 pts** Showed some steps to compute $$A^{-1}$$

part c out of 2 points

✓ **+ 2 pts** **Correct answer (48), with explanation**

- **+ 1 pts** Found det (A) instead
- **+ 0 pts** Incorrect (note $$det(2A)\neq 2det(A)$$)
- **+ 1 pts** Got wrong value for a or Det(A), but computed determinant correctly based on this value.

2. (6pts) (Note: you may use a calculator or other technology to check for errors in the problem, but you should be sure to show your work, so I can verify that you didn't exclusively use a calculator.)

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}$$

(a) Compute the singular values for A.

$$A^TA = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$$

$f_{A^TA}(\lambda) = \lambda^2 - 6\lambda = \lambda(\lambda - 6)$

$\lambda = 6, 0$

$\therefore \sigma$ of $A = \sqrt{6}, 0$

(b) Compute the SVD for A. In other words, compute orthogonal matrices U and V and a matrix Σ such that $A = U\Sigma V^T$.

$\Sigma = \begin{bmatrix} \sqrt{6} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ from part A

Find V: $E_{\sqrt{6}} = \ker\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$E_0 = \ker\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} = \text{Span}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$V^{*} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, Normalize V^{*},

$V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $V^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Finding U:

$$\frac{1}{\sqrt{6}}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2\sqrt{3}}\begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = U_1$$

Σ is 3x2 $\therefore U$ must be 3x3, U_2 & U_3 must $\perp U_1$

Test $V_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $U_1 \cdot V_2 = 0$ ✓, $U_1 \perp V_2$, $U_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Test $V_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ $U_1 \cdot V_3 = 0$ ✓, $U_2 \cdot V_3 = 0$ ✓, $U_1 \perp V_3$ & $U_2 \perp V_3$, $U_3 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

$$A = U\Sigma V^T = \underbrace{\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} \sqrt{6} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\Sigma}\underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}_{V^T}$$

OVER →

2 Question 2 **6 / 6**

✓ **+ 6 pts** **All correct.**

+ 1 pts Correct computation of singular values

+ 1 pts Correct computation of $$\Sigma$$ (based on sigular values from part a) Must be a 3x2 matix.

+ 2 pts Correct computation of V (don't forget to normalize)

+ 2 pts Correct computation of U(again this should be an orthogonal matrix)

+ 1 pts correctly computed some columns or aspects of U, but is missing some columns or did not make the columns O.N. Or some columns are incorrect

+ 0 pts See comments

+ 1 pts Parts of V correct

+ 1 pts Parts of Sigma correct

+ 0 pts No credit

3. (5 pts) Consider the following matrix A, which defines a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(\bar{x}) = A\bar{x}$.

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

(a) Given an independent list of vectors u, v in $\mathbb{R}^2$, is the list $T(\bar{u}), T(\bar{v})$ also independent. Justify your answer.

det(A) = 6, so A is an invertible transformation. Since A is invertible, that means that Ker(A) = $\vec{0}$, and no dimensions are collapsed to $\vec{0}$.

Since A is an invertible transformation, the linear transformation preserves linear independence.

Given a list of independent vectors u, v, T($\vec{u}$) and T($\vec{v}$) will also be independent

Ex. of linear transformation not preserving = projection onto line, but A is not projection onto line (dim(kerA) = 0)

(b) Given orthogonal vectors u, v in $\mathbb{R}^2$, are the vectors $T(\bar{u}), T(\bar{v})$ also orthogonal. Justify your answer.

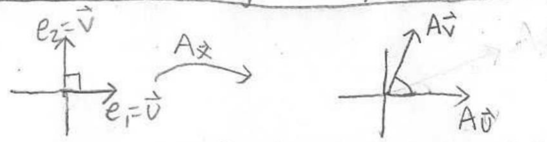
Counterexample. $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{u} \perp \vec{v}$

$T(\vec{u}) = A\vec{u} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$T(\vec{v}) = A\vec{v} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 2$, $2 \neq 0$, $T(\vec{u})$ is not $\perp$ $T(\vec{v})$

Using the elematory vectors as an example, we can see that the transformation does not preserve orthogonality.

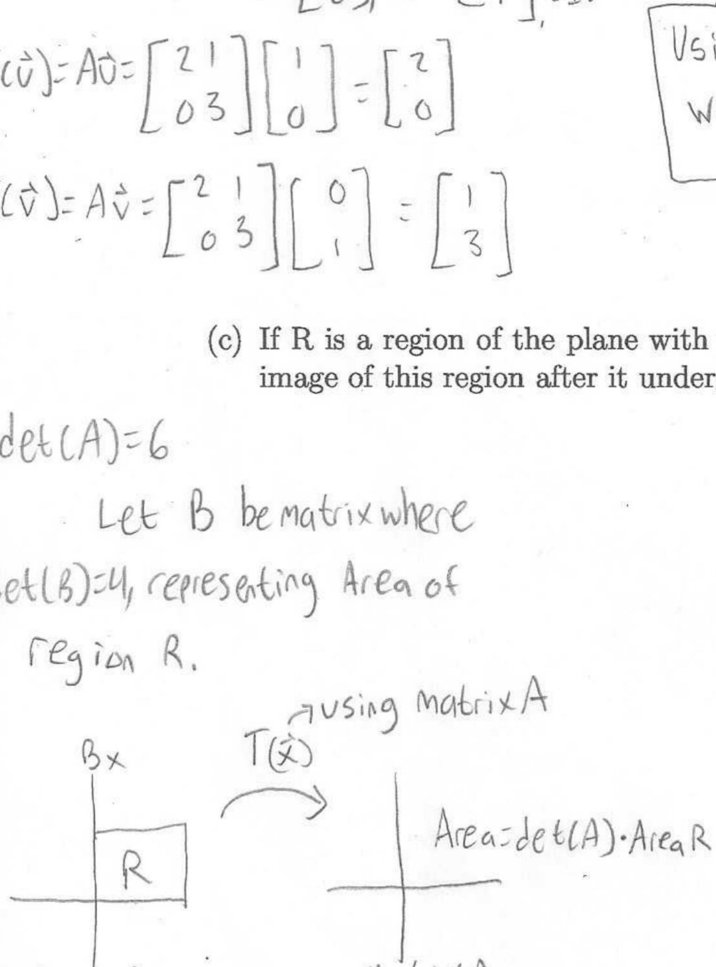


(c) If R is a region of the plane with area equal to 4 units², what is the area of the image of this region after it undergoes the transformation T?

det(A) = 6

Let B be matrix where det(B) = 4, representing Area of region R.

New Area = det A · Area (R) = 6 · 4 = 24 units²



OVER →

Since it's like finding composition

det(AB) = det(A ∘ B) = det(A) · det(B)

3 Question 3 **5 / 5**

✓ **+ 5 pts** **All correct**

+ **2 pts** Part a correct (Yes they are independent with justification)

+ **1 pts** Part a partially correct. (Yes they are independent but no justification or incorrect justification)

+ **2 pts** Part b correct (no they are no orthogonal with justification. Justification can be that A is not an orthogonal matrix, or give a counterexample)

+ **1 pts** Part b is partially correct (justification is not sufficient but the answer is correct)

+ **1 pts** Correct part c- with justification (i.e Area =4*det(A)=24

+ **0.75 pts** arithmetic error in part c, but otherwise correct.

+ **0 pts** All parts are wrong.

4. (4 pts) Find an example of a matrix that is invertible and has at least one eigenvalue, but is not diagonalizable. Be sure to explain why your given matrix has the desired properties.

Test: $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

λ of A = 1, almu of 3. ✓

det(A) = 1, so A is invertible ✓ because det(A) ≠ 0.

(easy to find determinant because A is upper triangular matrix)

($A^{-1} = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$ by technology as well)

Finding geomu (1).

$E_1 = \ker \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$

$\dim(E_1) = 1 \therefore \text{gemu}(1) = 1$

Since the gemu of all eigenvalues of A is 1 but A is 3x3, we do not have enough linearly independent vectors to form an eigen basis.

gemu (1) = 1, 1 < 3

Since we can't form an eigenbasis, there is no matrix S where we can compute $A = SBS^{-1}$ and have B be a diagonal matrix.

$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ is invertible, has an eigenvalue of 1, but is not diagonalizable.

OVER →

4 Question 4 **4 / 4**

- ✓ **+ 1 pts** Correct example
- ✓ **+ 1 pts** Justification that example is invertible
- ✓ **+ 1 pts** Justification that this example has at least one eigenvalue
- ✓ **+ 1 pts** Justification that this matrix is not diagoanlizable.
- **+ 0 pts** No answer

5. (8 pts) For each of the following descriptions of a square matrix below, identify it as either an invertible matrix, or not an invertible matrix. Justify your answer using a theorem that we have learned in our class.

(a) A 3×3 matrix whose rows are the vectors $\bar{v}, \bar{u}$ and $\bar{v} - \bar{u}$ ×

$A = \begin{bmatrix} \leftarrow v \rightarrow \\ \leftarrow u \rightarrow \\ \vec{v} - \vec{u} \end{bmatrix}$

$A^T = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ v & u & \vec{v}-\vec{u} \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$

We know that a matrix is invertible if its columns are linearly independent. $\vec{v} - \vec{u}$ is a linear combination of $\vec{v}$ and $\vec{u}$, so A^T is not invertible, since its columns are not linearly independent. Since A^T is not invertible, we know $\det(A^T) = 0$. Since $\det(A) = \det(A^T)$, $\det(A) = 0$ as well. Since $\det(A) = 0$, A is not an invertible matrix.

(b) A positive definite symmetric matrix. ✓

Positive Definite: $\lambda > 0$. By our characteristics of invertible matrices, a matrix is invertible if it does not have 0 as an eigenvalue. Since positive definite symmetric matrices only have eigenvalues > 0, 0 will never be an eigenvalue of the matrix, so a positive definite symmetric matrix is invertible.

(c) A matrix A which is similar to an orthogonal matrix B. ✓

Orthogonal matrices are invertible by their definition and $B^{-1} = B^T$. Orthogonal matrices also only have eigenvalues of 1 or -1. Since Matrix A is similar to Matrix B, A and B have the same eigenvalues. Since A and B can only have eigenvalues of 1 or -1 and a matrix is invertible if it doesn't have 0 as an eigenvalue, A is invertible.

(d) A matrix such that $A\bar{v} = A\bar{w}$ for some vectors $\bar{v} \neq \bar{w}$. ✓

A matrix is invertible if $A\vec{x} = \vec{b}$ has a unique solution for all vectors $\vec{b}$.

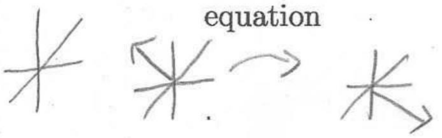
Let $A\vec{v} = \vec{b} = A\vec{w}$. Since $\vec{v}$ and $\vec{w}$ are both solutions to $A\vec{x} = \vec{b}$, then the Matrix A is not invertible, since $A\vec{x} = \vec{b}$ does not have a unique solution.

OVER →

5 Question 5 **8 / 8**

- ✓ **+ 2 pts** Part a) not invertible with justification
- ✓ **+ 2 pts** part b) invertible with justification
- ✓ **+ 2 pts** part c) invertible with justification
- ✓ **+ 2 pts** part d) not invertible with justification.
- **+ 0 pts** incorrect justification for one or more parts.

6. (9pts) Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a reflection through the plane P described by the equation



$$-x + 2y + z = 0$$

(a) Arguing geometrically, find the eigenvalues for the transformation T.

Eigenvalues mean that $Ax = \lambda x$. Since T is a reflection through the Plane P, any vector in plane P would be an eigenvector with eigenvalue 1 since it stays the same. Any vector that is orthogonal to the plane P would be flipped about the origin, so a vector orthogonal to the plane P would be an eigenvector with the eigenvalue -1.

The transformation T has eigenvalues of 1 and -1

(b) Find a basis for the each eigenspace of T.

Basis of E_1: 2 independent vectors that span plane: $-x+2y+z=0$

$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ $\quad -1+1=0$ ✓ $\quad -1+2-1=0$ ✓

Basis of E_1: $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

Basis of E_{-1}: vector perpendicular to plane, $\langle -1, 2, 1 \rangle$

Basis of E_{-1}: $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

Eigenbasis: $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$

(c) Suppose that A is the matrix such that $T(\bar{x}) = A\bar{x}$. Explain why how you know that this matrix must be symmetric, even without calculating it.

Since $\dim(E_1) + \dim(E_{-1}) = 3$ and A is a matrix in $\mathbb{R}^3$, we can form an eigenbasis for A. On top of that, we can form an orthogonal eigenbasis S for A such that $A = S^{-1}BS$, if we make our basis for E_1 two orthogonal unit vectors in the plane P, and make our basis for E_{-1} a unit vector. (it is also orthogonal to the vectors of the basis of E_1 since the basis vector for E_{-1} is in the orthogonal complement of the plane P.)

$\therefore$ A is orthogonally diagonalizable. By Spectral Theorem, we know that A is a symmetric matrix since A is orthogonally diagonalizable.

OVER →

6 Question 6 **9 / 9**

- ✓ **- 0 pts** correct
- **- 1 pts** part a one eigenvalue wrong
- **- 1 pts** part b missing vector or too many vectors or one vector wrong
- **- 1 pts** part c needs more explanation
- **- 3 pts** part b vectors wrong
- **- 3 pts** part c wrong
- **- 2 pts** part b 2 vectors wrong
- **- 3 pts** part a wrong
- **- 2 pts** part a two eigenvalue wrong
- **- 9 pts** not found

7. (8pts) There is a population of owls and squirrels living together in a forest. Let O_t be the population of owls at time t, and S_t be the population of squirrels at time t in thousands. Because the owls prey on the squirrels, both populations are dependent on the other. This can be modeled by the discrete dynamical system such that

$$\begin{bmatrix} O_{t+1} \\ S_{t+1} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.3 \\ -0.4 & 1.3 \end{bmatrix} \begin{bmatrix} O_t \\ S_t \end{bmatrix}$$

This coefficient matrix has eigenvalue 1.1 with eigenvector $\bar{v}_1 = (1, 2)$ and eigenvalue 0.7 with eigenvector $\bar{v}_2 = (3, 2)$.

Use this information to answer the following questions.

(a) Suppose that $\bar{x}_0$ is a vector representing the initial number of owls and squirrels and write $\bar{x}_0 = a\bar{v}_1 + b\bar{v}_2$. Compute $A^t\bar{x}_0$ in terms of a, b and t.

$\vec{x}_0 = c_1 v_1 + c_2 v_2$

$E_{1.1} = \text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$ $E_{0.7} = \text{span}\left\{\begin{bmatrix}3\\2\end{bmatrix}\right\}$

Eigenbasis: $\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ (v_1, v_2)

$$A^t(\vec{x}_0) = a(1.1)^t\begin{bmatrix}1\\2\end{bmatrix} + b(0.7)^t\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix} a(1.1)^t + 3b(0.7)^t \\ 2a(1.1)^t + 2b(0.7)^t \end{bmatrix}$$

(b) Compute $lim_{t\to\infty}A^t\bar{x}_0$ in terms of a and b.

$$\lim_{t\to\infty} A^t\vec{x}_0 = \lim_{t\to\infty}\begin{bmatrix} a(1.1)^t + 3b(0) \\ 2a(1.1)^t + 2b(0) \end{bmatrix} = \boxed{\infty\begin{bmatrix} a \\ 2a \end{bmatrix}}$$

(c) Use your answer from part c to describe the long term dynamics of this system. For what values of a and b will the owl and squirrel populations decline and eventually the animals will go extinct, and for what values of a and b will the populations continue to grow?

When $a=0$ and $b \geq 0$, the squirrel and owl populations will go extinct.
When $a>0$ and $b \geq 0$, the populations will continue to grow.
(assuming $a, b \geq 0$)

(d) Given that we start with an initial population where the animals do not go extinct, what will be the ratio between the number of owls and the number of squirrels in the long term?

Since $\begin{bmatrix}1\\2\end{bmatrix}$ is the corresponding eigenvector to $\lambda = 1.1$, the ratio of # owls to # of squirrels in the long term will be 1 owl to every 2 thousand squirrels.

Using our limit from part b

7 Question 7 **8 / 8**

- ✓ **+ 8 pts** **All correct**
- **+ 3 pts** Part a Correct
- **+ 1 pts** part b correct
- **+ 2 pts** part c correct
- **+ 2 pts** Part d correct.
- **+ 1.5 pts** part d) 1:2 instead of 1:2 thousand
- **+ 0 pts** No credit

Instructions:

- You should complete the exam and submit by 8am on June 7 PST. Please leave enough time to scan your work into a PDF and upload it to Gradescope! Exams will not be accepted after 8am.
- You may spend as much time as you like on this exam between now and the due date.
- If you do not have a way to print this exam or write directly on the PDF using a tablet, please copy each question onto a blank piece of paper. You final submission should include, the questions, your answers and any scratch work.
- This exam is open book. You can use any resources you find in our textbook, on our CCLE page or on the internet in general. However you should not have anyone's help to do the exam. So you are not allowed to ask your classmates or TAs about the questions or to post the exam questions on an online forum. Posting our exam questions online is the same asking someone to do the exam for you which is cheating.
- If you have a question about the phrasing of the one of the questions or about the mechanics of completing the exam, you can email ProfRosesMathExamQuestions@gmail.com.

By signing below, I certify that the following exam is entirely my own work. I did not receive any help in completing the exam. I will not post the exam questions on public or class forums or discuss the questions with anyone else until after the exam has ended.

Matthew Nieva

(Signature)

Remember to explain your calculations for full credit! It's fine to use technology to check your answer, but please write out some intermediate steps in your row reductions so we can see your process!

OVER →

8 Signature 3 / 3

✓ + 3 pts **Signature present.**