1. (7 pts) Suppose that the vector $\bar{v} = (a, -2)$ and the vector $\bar{w} = (2a, 9)$ are orthogonal vectors.

Also suppose that the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & a & 1 \\ 0 & 6 & a-1 \end{bmatrix}$$

is invertible.

(a) Use this information to solve for a.

$$\bar{v} \cdot \bar{w} = \begin{pmatrix} a \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2a \\ 9 \end{pmatrix} = 2a^2 - 18 = 0$$

$$2a^2 = 18$$

$$a^2 = 9$$

$$a = \pm 3$$

If $a = 3$

$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 6 & 2 \end{bmatrix}$ $\quad \mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 2/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{bmatrix} \neq I_3$

so $a \neq 3$

If $a = -3$

$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 6 & -4 \end{bmatrix}$ $\quad \mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$

so $a = -3$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 6 & 2 \end{bmatrix} \xrightarrow[\mathrm{III}-2\mathrm{II}]{\mathrm{I}-\frac{1}{3}\mathrm{II}} \begin{bmatrix} 1 & 0 & 2/3 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\div 3} \begin{bmatrix} 1 & 0 & 2/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 6 & -4 \end{bmatrix} \xrightarrow[\mathrm{III}+2\mathrm{II}]{\mathrm{I}+\frac{1}{3}\mathrm{II}} \begin{bmatrix} 1 & 0 & 4/3 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow[\mathrm{II}+\frac{1}{2}\mathrm{III}]{\mathrm{I}+\frac{2}{3}\mathrm{III}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Use your answer from part a) to compute the matrix A^{-1}. (Be sure to show your work)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 6 & -4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 & 1 & 0 \\ 0 & 6 & -4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\mathrm{III}+2\mathrm{II}]{\mathrm{I}+\frac{1}{3}\mathrm{II}} \left[\begin{array}{ccc|ccc} 1 & 0 & 4/3 & 1 & 1/3 & 0 \\ 0 & -3 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 & 2 & 1 \end{array}\right] \xrightarrow[\mathrm{II}+\frac{1}{2}\mathrm{III}]{\mathrm{I}+\frac{2}{3}\mathrm{III}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 5/3 & 2/3 \\ 0 & -3 & 0 & 0 & 2 & 1/2 \\ 0 & 0 & -2 & 0 & 2 & 1 \end{array}\right]$$

$$\xrightarrow[\mathrm{III}/-2]{\mathrm{II}/-3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 5/3 & 2/3 \\ 0 & 1 & 0 & 0 & -2/3 & -1/6 \\ 0 & 0 & 1 & 0 & -1 & -1/2 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 5/3 & 2/3 \\ 0 & -2/3 & -1/6 \\ 0 & -1 & -1/2 \end{bmatrix}$$

(c) Use your answer from part a) to compute the determinant of the matrix 2A.

$$2A = \begin{bmatrix} 2 & 2 & 2 \\ 0 & -6 & 2 \\ 0 & 12 & -8 \end{bmatrix}$$

$$\det(2A) = 2 \det \begin{bmatrix} -6 & 2 \\ 12 & -8 \end{bmatrix} = 2(6 \times 8 - 2 \times 12) = 48$$

OVER →

2. (6pts) (Note: you may use a calculator or other technology to check for errors in the problem, but you should be sure to show your work, so I can verify that you didn't exclusively use a calculator.)

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}$$

(a) Compute the singular values for A.

$A^T A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$

$f_A(\lambda) = \lambda^2 - 6\lambda = 0$

$\lambda(\lambda - 6) = 0$

$\lambda_1 = 6 \qquad \lambda_2 = 0$

$\sigma_1 = \sqrt{6} \qquad \sigma_2 = 0$

(b) Compute the SVD for A. In other words, compute orthogonal matrices U and V and a matrix Σ such that $A = U\Sigma V^T$.

$E_6 = \ker \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} = \operatorname{span} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad E_0 = \ker \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} = \operatorname{span} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\bar{u}_1 = \frac{1}{\sigma_1} A\bar{v}_1 = \frac{1}{\sqrt{2}\cdot\sqrt{6}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}$

Let $\bar{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \bar{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\bar{u}_2 = \frac{\bar{u}_2^\perp}{\|u_2\|} \quad \bar{u}_2^\perp = \bar{w}_1 - (\bar{u}_1 \cdot \bar{w}_1)\bar{u}_1 = \bar{u}_2 = \bar{w}_1 \quad \|\bar{u}_2\| = \sqrt{2}$

$\bar{u}_2 = \frac{\bar{w}_2}{\|\bar{w}_2\|} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$

$\bar{u}_3^\perp = \bar{u}_3 - (\bar{u}_1 \cdot \bar{u}_3)\bar{u}_1 - (\bar{u}_2 \cdot \bar{u}_3)\bar{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \\ 1/2 \end{bmatrix}$ OVER →

$\sqrt{\bar{u}_3^\perp} = \sqrt{\frac{1}{4} + 1 + \frac{1}{4}} = \frac{\sqrt{6}}{2} \qquad \bar{u}_3 = \frac{\bar{u}_3^\perp}{\|\bar{u}_3^\perp\|} = \frac{2}{\sqrt{6}} \begin{bmatrix} -1/2 \\ 1 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$

$\Sigma = \begin{bmatrix} \sqrt{6} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \quad U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$

$c_1 A\bar{u} + c_2 A\bar{v} = 0$
$A(c_1\bar{u} + c_2\bar{v}) = 0$
$A \neq 0$, so $c_1\bar{u} + c_2\bar{v} = 0$
$\bar{u}, \bar{v}$ are independent, so $c_1 = c_2 = 0$

3. (5 pts) Consider the following matrix A, which defines a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(\bar{x}) = A\bar{x}$.

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

(a) Given an independent list of vectors u, v in $\mathbb{R}^2$, is the list $T(\bar{u}), T(\bar{v})$ also independent. Justify your answer.

let $\bar{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ $\bar{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

~~If $T(\bar{u})$ $T(\bar{v})$ are independent~~

~~$c_1 T(\bar{u}) + c_2 T(\bar{v}) = 0$ when $c_1 c_2 \neq 0$~~ If $T(\bar{u}), T(\bar{v})$ are dependent, c_1 c_2 can be nonzero

$c_1 T(\bar{u}) + c_2 T(\bar{v}) = 0$

$c_1 \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + c_2 \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

$c_1 \begin{bmatrix} 2u_1 + u_2 \\ 3u_2 \end{bmatrix} + c_2 \begin{bmatrix} 2v_1 + v_2 \\ 3v_2 \end{bmatrix} = 0$

$c_1 \begin{bmatrix} 2u_1 + u_2 \\ 3u_2 \end{bmatrix} = -c_2 \begin{bmatrix} 2v_1 + v_2 \\ 3v_2 \end{bmatrix}$

$$\begin{cases} 2c_1u_1 + c_1u_2 = -2c_2v_1 - c_2v_2 \\ 3c_1u_2 = -3c_2v_2 \end{cases}$$

$2c_1u_1 + \cancel{c_1u_2} = -2c_2v_1 \cancel{+c_1u_2}$

$$\begin{cases} c_1u_1 = -c_2v_1 \\ c_1u_2 = -c_2v_2 \end{cases}$$

If $T(\bar{u}), T(\bar{v})$ are dependent, $\bar{u}, \bar{v}$ must be dependent, which contradicts with the fact that they are independent.
So $T(\bar{u}), T(\bar{v})$ are independent.

(b) Given orthogonal vectors u, v in $\mathbb{R}^2$, are the vectors $T(\bar{u}), T(\bar{v})$ also orthogonal. Justify your answer.

The vectors $T(\bar{u})$ and $T(\bar{v})$ are orthognal if A is orthognal
$A^T \neq A^{-1}$ so A is not orthogonal $\Rightarrow$ $T(\bar{u})$ $T(\bar{v})$ are not orthogonal.

Also, $T(\bar{u}) \cdot T(\bar{v}) = \begin{bmatrix} 2u_1 + u_2 \\ 3u_2 \end{bmatrix} \begin{bmatrix} 2v_1 + v_2 \\ 3v_2 \end{bmatrix} = 4u_1v_1 + 2u_1v_2 + 2u_2v_1 + 2u_2v_2 + 9u_2v_2$

$\bar{u} \cdot \bar{v} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = u_1v_1 + u_2v_2 = 0 \Rightarrow u_2v_2 = -u_1v_1$

$T(\bar{u}) \cdot T(\bar{v}) = -6u_1v_1 + 2u_1v_2 + 2u_2v_1$

Since $\|\bar{u}\| = 1$, $u_1^2 + u_2^2 = 1 \neq 0$

$T(\bar{u}) \cdot T(\bar{v}) \neq 0$, so $T(\bar{u}), T(\bar{v})$ not orthogonal

(c) If R is a region of the plane with area equal to 4 units^2, what is the area of the image of this region after it undergoes the transformation T?

$\det(A) = 2 \times 3 = 6$

Area after T = $\det(A)$ × original area $= 6 \times 4 = 24$ units^2

OVER →

4. (4 pts) Find an example of a matrix that is invertible and has at least one eigenvalue, but is not diagonalizable. Be sure to explain why your given matrix has the desired properties.

$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

A ~~in~~ is invertible because $\det(A) = 1 \neq 0$

A has eigenvalue 1 with $\mathrm{almu}(1) = 2$

$f_A(\lambda) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$

$E_1 = \ker \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \mathrm{span} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ with $\mathrm{geomu}(1) = 1 < 2$ (number of columns)

$\dim(E_1) = 1 < 2$

So A is not diagonalizable

OVER →

5. (8 pts) For each of the following descriptions of a square matrix below, identify it as either an invertible matrix, or not an invertible matrix. Justify your answer using a theorem that we have learned in our class.

(a) A 3×3 matrix whose rows are the vectors $\bar{v}, \bar{u}$ and $\bar{v} - \bar{u}$

$$A = \begin{bmatrix} - \bar{v} - \\ - \bar{u} - \\ - \bar{v} - \bar{u} - \end{bmatrix}$$

The third row of A is not linearly independent,

So rank $(A) < 3$, A is not invertible

(b) A positive definite symmetric matrix.

A is symmetric $\Rightarrow$ A is square matrix

A is positive definite $\Rightarrow$ all $\lambda > 0$

A is a square matrix and $\lambda \neq 0$, so A is invertible

(c) A matrix A which is similar to an orthogonal matrix B.

Let A, B be $n \times n$ matrices

B is orthogonal $\Rightarrow$ rank $(B) = n$

A, B are similar $\Rightarrow$ rank$(A) =$ rank $(B) = n$

So A is invertible

(d) A matrix such that $A\bar{v} = A\bar{w}$ for some vectors $\bar{v} \neq \bar{w}$.

$A\bar{v} = A\bar{w} = \bar{b}$, and $\bar{v} \neq \bar{w}$

$A\bar{x} = \bar{b}$ doesn't have a unique solution

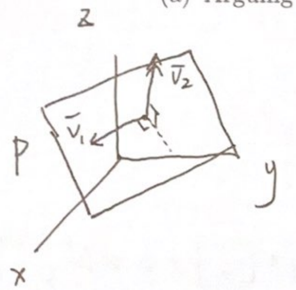
So A is not invertible

OVER →

6. (9pts) Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a reflection through the plane P described by the equation

$$-x + 2y + z = 0$$

(a) Arguing geometrically, find the eigenvalues for the transformation T.



① If $\bar{v}$ is normal to P ($\bar{v}_2$), It becomes $-\bar{v}$ after reflection T, so $\lambda_1 = -1$

② If $\bar{v}$ is in the plane P ($\bar{v}_1$), It remains unchanged, so $\lambda_2 = 1$ (algmu$(1) = 2$)

(b) Find a basis for the each eigenspace of T.

$E_{-1} = \text{span}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ (the line that is perpendicular to P)

$E_1 = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}\right\}$ (the plane P)

(c) Suppose that A is the matrix such that $T(\bar{x}) = A\bar{x}$. Explain why how you know that this matrix must be symmetric, even without calculating it.

A is a reflection matrix $\Rightarrow A^{-1} = A$

$\|A(\bar{x})\| = \|\bar{x}\|$ due to reflection $\Rightarrow A^T = A^{-1}$

So $A = A^{-1} = A^T$, A is symmetric

OVER →

7. (8pts) There is a population of owls and squirrels living together in a forest. Let O_t be the population of owls at time t, and S_t be the population of squirrels at time t in thousands. Because the owls prey on the squirrels, both populations are dependent on the other. This can be modeled by the discrete dynamical system such that

$$\begin{bmatrix} O_{t+1} \\ S_{t+1} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.3 \\ -0.4 & 1.3 \end{bmatrix} \begin{bmatrix} O_t \\ S_t \end{bmatrix}$$

This coefficient matrix has eigenvalue 1.1 with eigenvector $\bar{v}_1 = (1, 2)$ and eigenvalue 0.7 with eigenvector $\bar{v}_2 = (3, 2)$.

Use this information to answer the following questions.

(a) Suppose that $\bar{x}_0$ is a vector representing the initial number of owls and squirrels and write $\bar{x}_0 = a\bar{v}_1 + b\bar{v}_2$. Compute $A^t\bar{x}_0$ in terms of a, b and t.

$\cancel{\text{let } \bar{x}_0 =}$

$\bar{x}_0 = a\begin{bmatrix}1\\2\end{bmatrix} + b\begin{bmatrix}3\\2\end{bmatrix}$

$\bar{x}_0 = \begin{bmatrix}a+3b\\2a+2b\end{bmatrix}$

$A^t\bar{x}_0 = a\lambda_1^t\bar{v}_1 + b\lambda_2^t\bar{v}_2 = a\cdot 1.1^t\begin{bmatrix}1\\2\end{bmatrix} + b\cdot 0.7^t\begin{bmatrix}3\\2\end{bmatrix}$

$= \begin{bmatrix}1.1^t a\\2\times 1.1^t a\end{bmatrix} + \begin{bmatrix}0.7^t\cdot 3b\\0.7^t\cdot 2b\end{bmatrix}$

$A^t\bar{x}_0 = \begin{bmatrix}1.1^t a + 0.7^t\cdot 3b\\2\cdot 1.1^t a + 2\cdot 0.7^t 3b\end{bmatrix}$

(b) Compute $lim_{t\to\infty}A^t\bar{x}_0$ in terms of a and b.

$\lim_{t\to\infty} 0.7^t \to 0$, $\lim_{t\to\infty} A^t\bar{x}_0 = \begin{bmatrix}1.1^t a\\2\cdot 1.1^t a\end{bmatrix} = 1.1^t\begin{bmatrix}a\\2a\end{bmatrix} \to \infty$

(c) Use your answer from part c to describe the long term dynamics of this system. For what values of a and b will the owl and squirrel populations decline and eventually the animals will go extinct, and for what values of a and b will the populations continue to grow?

In the long term, both owl and squirrel population will continue to grow with the speed of 10% increase after each time period.

$\frac{O_t}{S_t} > \frac{3}{2}$

If $\cancel{> \frac{a}{b} > \frac{3}{2}}$, owls and squirrels will go extinct

~~If $0 \le \frac{a}{b} < \frac{3}{2}$, they will continue to grow~~ The population will continue to grow as long as a $\ne 0$

(d) Given that we start with an initial population where the animals do not go extinct, what will be the ratio between the number of owls and the number of squirrels in the long term?

$$\lim_{t\to\infty}\frac{O_t}{S_t} = \lim_{t\to\infty}\frac{1.1^t a}{2\cdot 1.1^t a} = \frac{1}{2}$$