Instructions:

- You should complete the exam and submit by 8am on June 7 PST. Please leave enough time to scan your work into a PDF and upload it to Gradescope! Exams will not be accepted after 8am.
- You may spend as much time as you like on this exam between now and the due date.
- If you do not have a way to print this exam or write directly on the PDF using a tablet, please copy each question onto a blank piece of paper. You final submission should include, the questions, your answers and any scratch work.
- This exam is open book. You can use any resources you find in our textbook, on our CCLE page or on the internet in general. However you should not have anyone's help to do the exam. So you are not allowed to ask your classmates or TAs about the questions or to post the exam questions on an online forum. Posting our exam questions online is the same asking someone to do the exam for you which is cheating.
- If you have a question about the phrasing of the one of the questions or about the mechanics of completing the exam, you can email ProfRosesMathExamQuestions@gmail.com.

By signing below, I certify that the following exam (including this written part and the part on CCLE) is entirely my own work. I did not receive any help in completing the exam. I will not post the exam questions on public or class forums or discuss the questions with anyone else until after the exam has ended.

Eric Tang

(Signature)

Remember to explain your calculations for full credit! It's fine to use technology to check your answer, but please write out some intermediate steps in your row reductions so we can see your process!

OVER →

1. (7 pts) Suppose that the vector $\bar{v} = (a, -2)$ and the vector $\bar{w} = (2a, 9)$ are orthogonal vectors.

Also suppose that the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & a & 1 \\ 0 & 6 & a-1 \end{bmatrix}$$

is invertible.

(a) Use this information to solve for a.

$\vec{v} \cdot \vec{w} = 0$, $2a^2 - 18 = 0$, $a^2 = 9$, $a = \pm 3$

$\det(A) \neq 0$. $\det(A) = a(a-1) - 6$

If $a = 3$, $\det(A) = 3 \cdot 2 - 6 = 0$

Therefore, $a = -3$

(b) Use your answer from part a) to compute the matrix A^{-1}. (Be sure to show your work)

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 6 & -4 \end{bmatrix}^{-1}$, $\text{rref}\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 & 1 & 0 \\ 0 & 6 & -4 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & \frac{5}{3} & \frac{2}{3} \\ 0 & 1 & 0 & 0 & -\frac{2}{3} & -\frac{1}{6} \\ 0 & 0 & 1 & 0 & -1 & -\frac{1}{2} \end{array}\right]$

$A^{-1} = \begin{bmatrix} 1 & \frac{5}{3} & \frac{2}{3} \\ 0 & -\frac{2}{3} & -\frac{1}{6} \\ 0 & -1 & -\frac{1}{2} \end{bmatrix}$

(c) Use your answer from part a) to compute the determinant of the matrix 2A.

$\det(2A) = 2^3 \det(A) = 8 \times [(-3)(-4) - 6] = 48$

$\det(2A) = 48$

OVER →

2. (6pts) (Note: you may use a calculator or other technology to check for errors in the problem, but you should be sure to show your work, so I can verify that you didn't exclusively use a calculator.)

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}$$

(a) Compute the singular values for A.

$$A^TA = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 3-\lambda & 3 \\ 3 & 3-\lambda \end{bmatrix}\right) = (\lambda-3)^2-9 \,. \quad (\lambda-3)^2-9=0, \ \lambda_1 = 6, \lambda_2 = 0$$

$$\sigma_1 = \sqrt{6}, \ \sigma_2 = 0$$

(b) Compute the SVD for A. In other words, compute orthogonal matrices U and V and a matrix Σ such that $A = U\Sigma V^T$.

$$\Sigma = \begin{bmatrix} \sqrt{6} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$E_6 = \ker\left(\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}\right) = \text{span}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$E_0 = \ker\left(\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}\right) = \text{span}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ \vec{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$u_1 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

u_2 has to be orthogonal to u_1, $u_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

u_3 has to be orthogonal to u_1, u_2, $u_3 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$$U = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}\begin{bmatrix} \sqrt{6} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}^T$$

OVER →

3. (5 pts) Consider the following matrix A, which defines a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(\bar{x}) = A\bar{x}$.

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

(a) Given an independent list of vectors u, v in $\mathbb{R}^2$, is the list $T(\bar{u}), T(\bar{v})$ also independent. Justify your answer.

Yes. A is invertible and injective, meaning $\ker(A) = \vec{0}$ only and A is one-to-one. Therefore, A preserves linear independence of u and V.

(b) Given orthogonal vectors u, v in $\mathbb{R}^2$, are the vectors $T(\bar{u}), T(\bar{v})$ also orthogonal. Justify your answer.

No, the columns of A is not an orthonormal basis, therefore A is not orthogonal. If A is not orthogonal, it's not guaranteed that orthogonal vectors $u, \vec{v}$ still remain orthogonal after the transformation.

For example, $\vec{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \end{bmatrix} \neq 0$

(c) If R is a region of the plane with area equal to 4 units^2, what is the area of the image of this region after it undergoes the transformation T?

$\det(A) = 2 \cdot 3 - 0 = 6$

$4 \cdot 6 = 24$ units

OVER →

4. (4 pts) Find an example of a matrix that is invertible and has at least one eigenvalue, but is not diagonalizable. Be sure to explain why your given matrix has the desired properties.

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, A is invertible because rank(A) = 2

A is an upper triangular matrix, $\lambda = 1$ with algmu (2)

$E_1 = \ker\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \text{span}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

E_1 has geometric multiplicity of 1, which is less than 2 (A is 2×2). Therefore, A is not diagonalizable.

OVER →

5. (8 pts) For each of the following descriptions of a square matrix below, identify it as either an invertible matrix, or not an invertible matrix. Justify your answer using a theorem that we have learned in our class.

(a) A 3×3 matrix whose rows are the vectors $\bar{v}, \bar{u}$ and $\bar{v} - \bar{u}$

The matrix is not invertible since the third row is dependent on the first two. The matrix has rank = 2, which is less than 3, the number of columns of A

(b) A positive definite symmetric matrix.

If the matrix A is symmetric, it's diagonalizable. If A is positive definite, it means that all eigenvalues of A are greater than 0. $\det(A) \neq 0$ since 0 is never an eigenvalue. A is invertible

(c) A matrix A which is similar to an orthogonal matrix B.

$SAS^{-1} = B$ where S is invertible if A and B are similar.
$A = S^{-1}BS$, $A^{-1} = S^{-1}B^{-1}S$, B^{-1} exists because $B^T = B^{-1}$ for an orthogonal matrix. Therefore, A^{-1} exists. A is invertible

(d) A matrix such that $A\bar{v} = A\bar{w}$ for some vectors $\bar{v} \neq \bar{w}$.

The matrix A is not invertible because there isn't an unique $\vec{x}$ that satisfies $A\vec{x} = \vec{b}$ for every $\vec{b}$

OVER →

6. (9pts) Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a reflection through the plane P described by the equation

$$-x + 2y + z = 0$$

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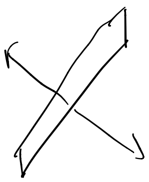

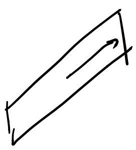

(a) Arguing geometrically, find the eigenvalues for the transformation T.

A eigenvalue exists when a vector remains parallel to itself after the transformation. In the reflection, only the vectors that are already on the plane and those perpendicular to the plane have this property. Vectors on the plane have $\lambda = 1$. Vectors perpendicular to the plane have $\lambda = -1$

(b) Find a basis for the each eigenspace of T.

$E_1 = \text{span}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$, $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ form a basis for E_1 and they span the plane.

$E_{-1} = \text{span} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ forms a basis for E_{-1} and it's perpendicular to the plane.

(c) Suppose that A is the matrix such that $T(\bar{x}) = A\bar{x}$. Explain why how you know that this matrix must be symmetric, even without calculating it.

A reflection matrix is orthogonally diagonalizable. And a matrix is orthogonally diagonalizable if and only if it's symmetric. Therefore, A is symmetric.

OVER →

7. (8pts) There is a population of owls and squirrels living together in a forest. Let O_t be the population of owls at time t, and S_t be the population of squirrels at time t in thousands. Because the owls prey on the squirrels, both populations are dependent on the other. This can be modeled by the discrete dynamical system such that

$$\begin{bmatrix} O_{t+1} \\ S_{t+1} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.3 \\ -0.4 & 1.3 \end{bmatrix} \begin{bmatrix} O_t \\ S_t \end{bmatrix}$$

This coefficient matrix has eigenvalue 1.1 with eigenvector $\bar{v}_1 = (1, 2)$ and eigenvalue 0.7 with eigenvector $\bar{v}_2 = (3, 2)$.

Use this information to answer the following questions.

(a) Suppose that $\bar{x}_0$ is a vector representing the initial number of owls and squirrels and write $\bar{x}_0 = a\bar{v}_1 + b\bar{v}_2$. Compute $A^t\bar{x}_0$ in terms of a, b and t.

$$A^t\vec{x_0} = 1.1^t a \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0.7^t b \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1.1^t a + 0.7^t(3b) \\ 1.1^t(2a) + 0.7^t(2b) \end{bmatrix}$$

(b) Compute $lim_{t\to\infty}A^t\bar{x}_0$ in terms of a and b.

$$\lim_{t\to\infty} A^t\vec{x_0} = \begin{bmatrix} \infty \\ 2\infty \end{bmatrix} a + \begin{bmatrix} 0 \\ 0 \end{bmatrix} b = \begin{bmatrix} \infty \\ 2\infty \end{bmatrix} a$$

(c) Use your answer from part c to describe the long term dynamics of this system. For what values of a and b will the owl and squirrel populations decline and eventually the animals will go extinct, and for what values of a and b will the populations continue to grow?

If $a = 0$ and $b > 0$, the owl and squirrel populations will decline and they eventually go extinct. If $a > 0$ and $b \in \mathbb{R}$, their populations will continue to grow.

(d) Given that we start with an initial population where the animals do not go extinct, what will be the ratio between the number of owls and the number of squirrels in the long term?

The ratio between the number of owls and squirrels in the long term is 1:2