1. (7 pts) Suppose that the vector $\bar{v} = (a, -2)$ and the vector $\bar{w} = (2a, 9)$ are orthogonal vectors.

Also suppose that the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & a & 1 \\ 0 & 6 & a-1 \end{bmatrix}$$

is invertible.

(a) Use this information to solve for a.

Invertible so $\det(A) \neq 0$.

$\bar{v}$ is orthogonal to $\bar{w}$ so $\bar{v} \cdot \bar{w} = 0$ $\quad \bar{v} \cdot \bar{w} = 2a^2 + (-18) = 0 \qquad 2a^2 = 18 \quad a^2 = 9 \quad a = \pm 3$

$$A = \left[\begin{array}{ccc|cc} 1 & 1 & 1 & 1 & 1 \\ 0 & a & 1 & 0 & a \\ 0 & 6 & a-1 & 0 & 6 \end{array}\right] \qquad \det(A) = a(a-1) + 0 + 0 - 0 - 6$$

Check: cofactor expansion along column 1

$$(-1)^{1+1}(1)\begin{vmatrix} a & 1 \\ 6 & a-1 \end{vmatrix} = a(a-1) - 6$$

$$\det(A) = a(a-1) - 6$$

If $a = 3$

$\det(A) = 3(3-1) - 6 = 3(2) - 6 = 0$ (A not invertible)

If $a = -3$

$\det(A) = -3(-3-1) - 6 = -3(-4) - 6 = 12 - 6 = \boxed{6 \neq 0}$ (so A is invertible)

$\boxed{a = -3}$

(b) Use your answer from part a) to compute the matrix A^{-1}. (Be sure to show your work)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 6 & -3-1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 6 & -4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & y_1 \\ 0 & -3 & 1 & y_2 \\ 0 & 6 & -4 & y_3 \end{array}\right] \xrightarrow{\div -3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & y_1 \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{3}y_2 \\ 0 & 6 & -4 & y_3 \end{array}\right] \xrightarrow[-6\mathrm{II}]{-\mathrm{II}} \left[\begin{array}{ccc|c} 1 & 0 & \frac{4}{3} & y_1 \ \frac{1}{3}y_2 \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{3}y_2 \\ 0 & 0 & -2 & 2y_2 \ y_3 \end{array}\right] \xrightarrow{\div -2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{4}{3} & y_1 \ \frac{1}{3}y_2 \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{3}y_2 \\ 0 & 0 & 1 & -y_2 \ -\frac{1}{2}y_3 \end{array}\right] \xrightarrow[+\frac{1}{3}\mathrm{III}]{-\frac{4}{3}\mathrm{III}} \left[\begin{array}{ccc|c} \multicolumn{3}{c|}{I_3} & A^{-1} \\ 1 & 0 & 0 & y_1 \ \frac{5}{3}y_2 \ \frac{4}{6}y_3 = \frac{2}{3}y_3 \\ 0 & 1 & 0 & -\frac{2}{3}y_2 \ -\frac{1}{6}y_3 \\ 0 & 0 & 1 & -y_2 \ -\frac{1}{2}y_3 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} 1 & \frac{5}{3} & \frac{2}{3} \\ 0 & -\frac{2}{3} & -\frac{1}{6} \\ 0 & -1 & -\frac{1}{2} \end{bmatrix}}$$

(c) Use your answer from part a) to compute the determinant of the matrix 2A.

$2A = 2\times$ every row of A

$$\begin{bmatrix} 2 & 2 & 2 \\ 0 & 2(-3) & 2(1) \\ 0 & 2(6) & 2(-4) \end{bmatrix} \begin{array}{c} \xrightarrow{\div 2} \\ \xrightarrow{\div 2} \\ \xrightarrow{\div 2} \end{array} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 6 & -4 \end{bmatrix}$$

$\boxed{\det(2A) = 48}$

$\det(2A) \longrightarrow \det(A)$

$\det(2A) \xleftarrow{\times 2} 6$

48

$\times 2$

$\times 2 \quad (\times 2^3 = 8)$

$6 \times 8 = 48$

OVER →

2. (6pts) (Note: you may use a calculator or other technology to check for errors in the problem, but you should be sure to show your work, so I can verify that you didn't exclusively use a calculator.)

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}$$

(a) Compute the singular values for A.

$A^TA = \underbrace{\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}}_{A} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$

characteristic polynomial of A^TA

for 2×2: $\lambda^2 - tr(A)\lambda + det(A)$

$f_{A^TA}(\lambda) = \lambda^2 - (3+3)\lambda + (9-9) = 0$

$= \lambda^2 - 6\lambda + 0 = 0$

$\lambda(\lambda - 6) = 0$

$\lambda_1 = 6 \quad \lambda_2 = 0$

Singular value $= \sqrt{\text{eigenvalues}}$ of A^TA

$\boxed{\sigma_1 = \sqrt{6} \quad \sigma_2 = \sqrt{0} = 0}$

(b) Compute the SVD for A. In other words, compute orthogonal matrices U and V and a matrix Σ such that $A = U\Sigma V^T$.

$3\times2 = 3\times3 \cdot 3\times2 \cdot 2\times2$

$\lambda_1 = 6 \quad \lambda_2 = 0$

$E_6 = ker(A^TA - 6I_2) = ker\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} : \left[\begin{array}{cc|c} -3 & 3 & 0 \\ 3 & -3 & 0 \end{array}\right] \xrightarrow{\substack{I/-3 \\ II/3}} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right] \xrightarrow{-I} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x_1 = x_2$, $x_2 = t$ $E_6 = span\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$E_0 = ker(A^TA) = ker\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \xrightarrow{\substack{I/3 \\ II/3}} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \xrightarrow{-I} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $E_0 = span\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Σ = same dim(A) = 3×2 $\begin{bmatrix} \sqrt{6} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

V = orthonormal basis 2×2

$\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\vec{v}_2$ = another orthonormal vector orthogonal to v_1 $= \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ $V^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

U = orthonormal basis 3×3

$\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{12}}\begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$

$\sqrt{12} = \sqrt{4\cdot3} = 2\sqrt{3}$

$U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$

Find 2 other orthogonal vectors by inspection / Gram Schmidt

$\vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\vec{u}_2 \cdot \vec{u}_1 = 0$, $\|\vec{u}_2\| = 1$

OVER →

$\vec{u}_3 = \dfrac{v_3^{\perp}}{\|v_3^{\perp}\|}$ $v_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$v_3 - (v_3 \cdot u_1)u_1 - (v_3 \cdot u_2)u_2$

$v_3^{\perp} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}\right)\frac{1}{\sqrt{12}}\begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right)\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 0_3 - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$\|v_3^{\perp}\| = \frac{1}{\sqrt{1+1+4}} = \frac{1}{\sqrt{6}}$ $\vec{u}_3 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$\boxed{A = U\Sigma V^T = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}\begin{bmatrix} \sqrt{6} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}$

3. (5 pts) Consider the following matrix A, which defines a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(\bar{x}) = A\bar{x}$.

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

(a) Given an independent list of vectors u, v in $\mathbb{R}^2$, is the list $T(\bar{u}), T(\bar{v})$ also independent. Justify your answer.

$\boxed{\text{YES}}$ $\bar{u}$ and $\bar{v}$ are independent, so $c_1\bar{u} + c_2\bar{v} = 0$ where $c_1, c_2 = 0$ only (only trivial solution).

so $A(c_1\bar{u} + c_2\bar{v}) = A(\bar{0}) = \bar{0}$ (trivial solution) is eqral to

$A(c_1\bar{u}) + A(c_2\bar{v}) = c_1 A(\bar{u}) + c_2 A(\bar{v}) = \bar{0}$ when $c_1\ c_2 = 0$.

$\det(A) = 6 \neq 0$

$\ker(A) = \left[\begin{array}{cc:c} 2 & 1 & 0 \\ 0 & 3 & 0 \end{array}\right] \begin{array}{c} \div 2 \\ \div 3 \end{array} \left[\begin{array}{cc:c} 1 & 1/2 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{-\frac{1}{2}II} \left[\begin{array}{cc:c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$ so $\ker(A) = \{\bar{0}\}$. A is invertible.

So only one vector $(\bar{0})$ maps to $\bar{0}$ so $c_1 A(\bar{v}) + c_2 A(\bar{v}) = 0$ only when $c_1\ c_2 = 0$ (trivial solution).

A is diagonalizable: $V = [\vec{u}\ \vec{v}]$ ($\lambda = 2, 3$, 2×2 matrix) $AV = SBS^{-1}V$ = transform basis to another basis through S^{-1}, apply stretch by 2 and 3, transform basis to another basis through S, so $A\bar{v}$ forms new basis.

(b) Given orthogonal vectors u, v in $\mathbb{R}^2$, are the vectors $T(\bar{u}), T(\bar{v})$ also orthogonal. Justify your answer.

$\boxed{\text{NO}}$, since A is not symmetric or orthogonally diagonalizable, so $\vec{u}$ and $\vec{v}$ do not have to be eigenvectors of A (where they are orthogonal after transformation).

$A^T = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \neq A$

EX: $\vec{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $A\vec{e_1} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ $\vec{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $A\vec{e_2} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $A\vec{e_1} \cdot A\vec{e_2} = 2 \neq 0$.

(c) If R is a region of the plane with area equal to 4 units^2, what is the area of the image of this region after it undergoes the transformation T?

$|\det(A)| = \dfrac{\text{area of } T(\Omega)}{\text{area of } \Omega}$

area of Ω $(\det(A))$ = area of $T(\Omega)$

$\det(A) = \overset{2(3)-0}{6}$

$4(6) = 24$ = new area under T

$\boxed{24 \text{ units}^2}$

OVER →

4. (4 pts) Find an example of a matrix that is invertible and has at least one eigenvalue, but is not diagonalizable. Be sure to explain why your given matrix has the desired properties.

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\det(A) = 1 \neq 0$ so A is invertible

$f_A(\lambda) = \lambda^2 - \underset{\text{tr}(A)}{2}\lambda + \underset{\det(A)}{1} = 0$ (triangular matrix so $\lambda = 1$ along diagonal)

$(\lambda - 1)(\lambda - 1)$ $\lambda = 1$ with $\text{agmu}(1) = 2$

$E_1 = \ker(A - I_2) = \ker\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $E_1 = \text{span}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Since $\text{geomu}(1) = 1$ and there is only one eigenvalue, the sum of the geomu $\neq 2$ so the matrix is not diagonalizable.

$\dim(E_1) = 1 \neq 2$ $s \neq n$.

Since the matrix is a shear, there is only one eigenvector and other vectors are stretched, so although A is invertible, it does not have enough dimensions for the eigenspaces.

OVER →

5. (8 pts) For each of the following descriptions of a square matrix below, identify it as either an invertible matrix, or not an invertible matrix. Justify your answer using a theorem that we have learned in our class.

(a) A 3×3 matrix whose rows are the vectors $\bar{v}, \bar{u}$ and $\bar{v} - \bar{u}$

$A = \begin{bmatrix} - \bar{v} - \\ - \bar{u} - \\ - \bar{v} - \bar{u} - \end{bmatrix}$

Since the row with $\bar{v}-\bar{u}$ can be written as a linearcombination of the previous rows, ($\bar{w} = 1\bar{v} - 1\bar{u}$ $c_1 = 1$ $c_2 = -1$), the row is redundant so the matrix is not invertible.

(b) A positive definite symmetric matrix.

A positive definise symmetric square matrix has eigenvalues all >0

So since no eigenvalues $= 0$, $\ker(A - \lambda I) \neq \ker(A)$ for all λ so $\ker(A) = \{\vec{0}\}$ (no eigenvector in kernel) and so A is invertible.

(c) A matrix A which is similar to an orthogonal matrix B.

Since A is similar to B, A and B must have the same eigenvalues,

Since B is an orthogonal transformation matrix, it preserves length of vectors

So $\|B\vec{x}\| = \|\vec{x}\| = \|\lambda\vec{x}\| = |\lambda|\|\vec{x}\|$ so $\lambda = \pm 1$. Since $\lambda \neq 0$ for all eigenvalues, A must be invertible.

(d) A matrix such that $A\bar{v} = A\bar{w}$ for some vectors $\bar{v} \neq \bar{w}$.

A is not invertible since $A\bar{x} = \bar{b}$ where $\bar{b} = A\bar{v} = A\bar{w}$ does not have an unique solution. $\bar{x}$ can be $\bar{v}$ or $\bar{w}$ where $\bar{v}$ and $\bar{w}$ are distinct vectors.

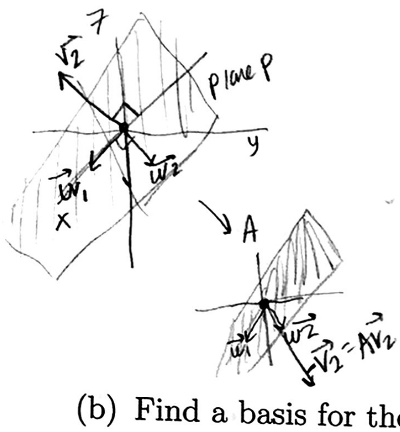
OVER →

6. (9pts) Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a reflection through the plane P described by the equation

$$-x + 2y + z = 0$$

.

(a) Arguing geometrically, find the eigenvalues for the transformation T.



Any eigenvector on the plane will stay on the plane after a reflection so $\lambda = 1$ (no change since already on the plane).

Any nonzero eigenvector perpendicular to the plane will reflect through the plane (so its direction is opposite but magnitude the same) $A\vec{v} = -\vec{v}$ so $\lambda = -1$.

(b) Find a basis for the each eigenspace of T.

E_1 = span (plane) or any 2 non parallel vectors on the plane

When $y=0$ $x=1, z=-1$ $\vec{w}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

When $y=1$ $x=0$ $z=-2$ $\vec{w}_2 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$

$E_1 = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \right\}$

E_{-1} = span (line) vector normal to the plane

normal vector $\vec{v}$: $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ coefficients of $-x + 2y + z = 0$

$E_{-1} = \text{span} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

(c) Suppose that A is the matrix such that $T(\bar{x}) = A\bar{x}$. Explain why how you know that this matrix must be symmetric, even without calculating it.

Since the transformation is a reflection, it is orthogonally diagonalizable, since after transformation the eigenvectors are still orthogonal to each other, (vectors in E_1 still orthogonal to vectors in E_{-1}), the eigenvectors of A are also eigenvectors of A^TA and A is orthoganally diagonalizable which implies A is symmetric (spectral theorem).

OVER →

7. (8pts) There is a population of owls and squirrels living together in a forest. Let O_t be the population of owls at time t, and S_t be the population of squirrels at time t in thousands. Because the owls prey on the squirrels, both populations are dependent on the other. This can be modeled by the discrete dynamical system such that

$$\begin{bmatrix} O_{t+1} \\ S_{t+1} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.3 \\ -0.4 & 1.3 \end{bmatrix} \begin{bmatrix} O_t \\ S_t \end{bmatrix}$$

This coefficient matrix has eigenvalue 1.1 with eigenvector $\bar{v}_1 = (1, 2)$ and eigenvalue 0.7 with eigenvector $\bar{v}_2 = (3, 2)$.

Use this information to answer the following questions.

(a) Suppose that $\bar{x}_0$ is a vector representing the initial number of owls and squirrels and write $\bar{x}_0 = a\bar{v}_1 + b\bar{v}_2$. Compute $A^t\bar{x}_0$ in terms of a, b and t.

$A^t\vec{x_0} = A^t(a\bar{v_1}+b\bar{v_2}) = A^t a\bar{v_1} + A^t b\bar{v_2}$
$= aA^t\bar{v_1} + bA^t\bar{v_2}$
$= a\lambda_1^t\bar{v_1} + b\lambda_2^t\bar{v_2}$
$= a(1.1)^t\bar{v_1} + b(0.7)^t\bar{v_2}$

$A^t\vec{x_0} = a(1.1)^t\begin{bmatrix}1\\2\end{bmatrix} + b(0.7)^t\begin{bmatrix}3\\2\end{bmatrix}$

$A^t\vec{x_0} = \begin{bmatrix} a(1.1)^t + 3b(0.7)^t \\ 2a(1.1)^t + 2b(0.7)^t \end{bmatrix}$

(b) Compute $lim_{t\to\infty}A^t\bar{x}_0$ in terms of a and b.

$\lim_{t\to\infty} A^t\vec{x_0} = \vec{x}_{equ}$

$\lim_{t\to\infty} A^t\vec{x_0} = a(1.1)^t\begin{bmatrix}1\\2\end{bmatrix} + \cancel{b(0.7)^t\begin{bmatrix}3\\2\end{bmatrix}}$

$= a(1.1)^t\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}a(1.1)^t\\2a(1.1)^t\end{bmatrix} \approx \begin{bmatrix}a\\2a\end{bmatrix}$ or $\begin{bmatrix}\infty\\\infty\end{bmatrix}$

close to 1

if 1.1 not approximated as 1

(c) Use your answer from part c to describe the long term dynamics of this system. For what values of a and b will the owl and squirrel populations decline and eventually the animals will go extinct, and for what values of a and b will the populations continue to grow?

$\lim_{t\to\infty} a(1.1)^t\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}\infty\\\infty\end{bmatrix}$.

For any a and b the populations will continue to grow since No value of a and b will cause extinction according to the formula other than $a=0$.

(d) Given that we start with an initial population where the animals do not go extinct, what will be the ratio between the number of owls and the number of squirrels in the long term?

In the long term, the population will almost be a multiple of (1,2) So the ratio will be approximately 1 owl for every 2 thousand squirrels and both grow by approximately 10% each year.

ratio $= \frac{1 \text{ owl}}{2000 \text{ squirrels}}$