Catherine Hu

Instructions:

- This exam is in two parts.
  - Part 1 is on CCLE- This is a multiple choice "quiz" that will be graded by the computer, similar to our pre-class assignments.
  - Part 2 is in the following pages and should be written by hand and uploaded to Gradescope.
- You should complete the exam and submit by 8am on Dec 14th PST. Please leave enough time to scan your work into a PDF and upload it to Gradescope! Exams will not be accepted after 8am.
- You may spend as much time as you like on this exam between now and the due date.
- If you do not have a way to print this exam you can copy each question onto a blank piece of paper. BEWARE- Please copy the entire question, and give yourself plenty of space to answer each question. We will take off points if, for example, your answer says

  2) [your answer]

  without including the phrasing of the question 2.
- This exam is open book. You can use any resources you find in our textbook, on our CCLE page or on the internet in general. However you should not have anyone's help to do the exam. So you are not allowed to ask your classmates or TAs about the questions or to post the exam questions on an online forum. Posting our exam questions online is the same asking someone to do the exam for you which is cheating.
- If you have a question about the phrasing of the one of the questions or about the mechanics of completing the exam, you can email ProfRosesMathExamQuestions@gmail.com.

By signing below, I certify that the following exam (including this written part and the part on CCLE) is entirely my own work. I did not receive any help in completing the exam. I will not post the exam questions on public or class forums or discuss the questions with anyone else until after the exam has ended.

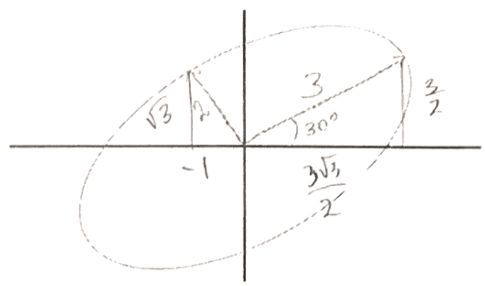
Cath

(Signature)

Remember to explain your calculations for full credit! It's fine to use technology to check your answer, but please write out some intermediate steps in your row reductions so we can see your process!

**OVER →**

1. (6 pts) In this class we learned that every $2 \times 2$ matrix takes a circle to an ellipse, and also how to find the ellipse using SVD. Now we'll do the reverse. Consider the ellipse pictured below, with semimajor axis of length 3 and at a 30 degree angle with the x-axis, and with and semiminor axis with length 2.




(a) Find a matrix A, that represents a transformation that takes the unit circle to this ellipse. Hint: You may want to think of your transformation as a composition of rotations and scaling.

check:
$e_1 \to T(e_1): \begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} \frac{3\sqrt{3}}{2} \\ \frac{3}{2} \end{bmatrix}$
$e_2 \to T(e_2): \begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix}$

$$\underbrace{\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}}_{\Sigma}\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{V^T} = \boxed{A = \begin{bmatrix} \frac{3\sqrt{3}}{2} & -1 \\ \frac{3}{2} & \sqrt{3} \end{bmatrix}}$$

rotates counterclockwise by 30°

scales x by 3, y by 2

(b) Use the determinant to compute the area of the ellipse.

$$\text{area} = |\det A|\,(\text{area of unit circle})$$
$$= \left[\left(\tfrac{3\sqrt{3}}{2}\right)\sqrt{3} + \tfrac{3}{2}\right]\pi$$
$$= \boxed{6\pi \text{ units}^2}$$

(c) If R is a region of the plane with area equal to 7 units$^2$, what is the area of the image of this region after it undergoes the transformation represented by the matrix $2A$?

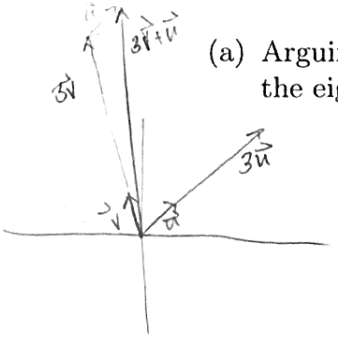
$$\text{image Area} = |\det(2A)| \cdot \underbrace{\text{region}}_{7}$$
$$= (2^2 \det A)(7)$$
$$= (4)(6)(7)$$
$$= \boxed{168 \text{ units}^2}$$

**OVER →**

$\to \{\vec{u}, \vec{v}\}$

2. (8 pts) Suppose that the vectors $\bar{u}$ and $\bar{v}$ are a basis $\mathcal{B}$ for $\mathbb{R}^2$.

Suppose further that T is a transformation $\mathbb{R}^2 \to \mathbb{R}^2$ that maps $\bar{u} \mapsto 3\bar{u}$, and $\bar{v} \mapsto \bar{u}+3\bar{v}$. Consider the matrix $A$ such that $T(\bar{x}) = A\bar{x}$.




(a) Arguing geometrically, identify one eigenvalue $\lambda$ for the matrix A, and describe the eigenspace in terms of $\bar{u}$ and $\bar{v}$. What is the geometric multiplicity of $\lambda$?

$\boxed{\lambda = 3}$ b/c $A\vec{u} = 3\vec{u}$

The eigenspace is spanned by $\vec{u}$. $\vec{v}$ is not an eigenvector in $E_3$ because: $A\vec{v} = \lambda\vec{v}$ where $\lambda$ is a scalar fails to be true.

Therefore, the $\boxed{\text{geometric multiplicity is } 1}$

(b) Find a matrix $B$ which is the $\mathcal{B}$ matrix for $T$.

The $\mathcal{B}$ matrix of T is: $B = \left[ [T(\vec{u})]_{\mathcal{B}}, [T(\vec{v})]_{\mathcal{B}} \right]$

$[T(\vec{u})]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}_{\mathcal{B}}$ $[T(\vec{v})]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}_{\mathcal{B}}$

$$\boxed{B = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}}$$

(c) What is the relationship between the eigenvalues of A and the eigenvalues of B? Use your answer to part b, to explain why the matrix A has only one eigenvalue.

the eigenvalues of A and B are the $\boxed{\text{same}}$, and the eigenvalue 3 in B has an algebraic multiplicity of 2, which explains why this is the only eigenvalue found in A

(d) What can you say about the algebraic multiplicity of the eigenvalue $\lambda$ from part a). Is the matrix A diagonalizable?

Because this is in $\mathbb{R}^2$, the algebraic multiplicity will be 2. Since its algebraic multiplicity does not equal its geometric multiplicity, A is $\boxed{\text{not diagonalizable.}}$

OVER →

3. (8 pts) For each of the following descriptions of a square matrix below, identify it as either an invertible matrix, or not an invertible matrix. Justify your answer using a theorem that we have learned in our class.

(a) A $3 \times 3$ matrix whose columns are the vectors $\bar{v}, \bar{u}$ and $2\bar{v} - \bar{u}$

$$A = \begin{bmatrix} | & | & | \\ \vec{v} & \vec{u} & 2\vec{u}-\vec{v} \\ | & | & | \end{bmatrix}$$

$\boxed{\text{not invertible}}$ - 3rd column ⇒ $2\vec{u}-\vec{v}$ is a linear combination of the first 2, meaning rank A < # of columns. To be an invertible square matrix in this case, rank A must equal 3, the # of columns, according to the main theorem for invertibility

(b) The following matrix, where $a$ is nonzero

yes, $\boxed{\text{it is invertible}}$

let $A = \begin{bmatrix} 0 & a & a \\ a & 0 & a \\ a & a & 0 \end{bmatrix}$

$$\det A = \begin{bmatrix} 0 & a & a \\ a & 0 & a \\ a & a & 0 \end{bmatrix} \begin{matrix} 0 & a \\ a & 0 \\ a & a \end{matrix} = 0 + a^3 + a^3 - 0 - 0 - 0 \quad \text{(Sarrus's Rule)}$$

$= 2a^3$ a is nonzero so determinant is nonzero

nonzero determinant means the matrix is invertible as well, also according to the main theorem for invertibility

(c) A matrix $A$ which is similar to an orthogonal matrix $B$. → $\boxed{\text{yes, it is invertible}}$

$AS = SB$ for some invertible matrix $S$

$\det(AS) = \det(SB)$

$\det A \det S = \det S \det B$

$\det A = \det B$

according to a ch. 6 Theorem, orthogonal matrix determinant = 1 or -1, so det B is nonzero and det A will be nonzero as well. Therefore, A is invertible.

(d) A $3 \times 3$ matrix representing a transformation T, where T takes a sphere in $\mathbb{R}^3$ to an ellipse in a plane inside $\mathbb{R}^3$. $T(\vec{x}) = A\vec{x}$

$\boxed{\text{not invertible}}$ — the dim (Im A) = 2, which is less than 3, so the meaning ker(A) is nontrivial. since ker(A) ≠ {0}, it's nontrivial so the matrix is not invertible because the matrix does not have full rank according to the main theorem for invertibility

**OVER →**

4. (6 pts) Find an example of a matrix with each of the following properties, and explain briefly (1-2 sentences), why the matrix you chose has the desired property. If no such matrix exists, use one of our theorems from class to explain why this can't happen.

(a) A square matrix which represents a transformation that is injective but not surjective. matrix A of size $n \times n$

injective: one to one, each x maps to a unique y → this means $\dim(\ker A) = 0$

surjective: each y has a corresponding x (image represents the full range)

with a square matrix, it is $\boxed{\text{impossible}}$. In order to be injective, the matrix must have full rank. However, because this is square, the full rank means the dim(im A) = rank and this is an invertible matrix (main theorem for invertibility), so it must be surjective as well

(b) A matrix that is invertible and has at least one eigenvalue, but is not diagonalizable.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

eigenvalue $\lambda = 1$, almu(1): 2 ⇒ 1 eigenvalue

det(A): 1 - 0 = 1 ≠ 0 ⇒ invertible ✓

$E_1 = \ker \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \text{span} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ geomu(1) = 1 ⇒ not diagonalizable

geomu(1) represents # of dimensions of the kernel — since it's less than rank of A, this matrix is not diagonalizable. However, it has a nonzero determinant, so it is invertible

(c) A matrix whose characteristic polynomial is $p(\lambda) = -\lambda(4 - \lambda)(2 - \lambda)^2$

$0 = -\lambda(4-\lambda)(2-\lambda)^2$

$\lambda = 0, 4, 2$

almu(0) = 1 almu(4) = 1 almu(2): 2

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

eigenvalues are represented along the diagonal of a diagonal matrix

OVER →

5. (6pts) Consider the matrix

$$A = \begin{bmatrix} 11 & -5 \\ -2 & 10 \end{bmatrix}$$

(a) Compute the singular values for A

$A^T = \begin{bmatrix} 11 & -2 \\ -5 & 10 \end{bmatrix}$ $\quad A^TA = \begin{bmatrix} 11 & -2 \\ -5 & 10 \end{bmatrix}\begin{bmatrix} 11 & -5 \\ -2 & 10 \end{bmatrix} = \begin{bmatrix} 125 & -75 \\ -75 & 125 \end{bmatrix}$

$f(\lambda) = \lambda^2 - 250\lambda + 10000 = 0$

$(\lambda - 50)(\lambda - 200) = 0 \Rightarrow \lambda_1 = 200 \quad \lambda_2 = 50$

$\sigma_1 = \sqrt{200} = 10\sqrt{2} \qquad \sigma_2 = \sqrt{50} = 5\sqrt{2}$

(b) Compute the SVD for A

$E_{200} = \ker\begin{bmatrix} -75 & -75 \\ -75 & -75 \end{bmatrix} = \operatorname{span}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad v_1 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

$E_{50} = \ker\begin{bmatrix} 75 & -75 \\ -75 & 75 \end{bmatrix} = \operatorname{span}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

$u_1 = \frac{1}{\sigma_1} Av_1 = \frac{1}{10\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 11 & -5 \\ -2 & 10 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{20}\begin{bmatrix} -16 \\ 12 \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$

$u_2 = \frac{1}{\sigma_2} Av_2 = \frac{1}{5\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 11 & -5 \\ -2 & 10 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$

$\boxed{V = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \quad \Sigma = \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix} \quad U = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}}$

(c) Given the equation $A = U\Sigma V^T$, prove that, if A is invertible, then $A^{-1} = V\Sigma^{-1}U^T$.

$U^{-1} = U^T$ & $V^{-1} = V^T$ b/c $U$ & $V$ are orthogonal

$A = U\Sigma V^{-1}$

$AA^{-1} = A^{-1}U\Sigma V^{-1}$

$I_n V = A^{-1} U \Sigma$

$V\Sigma^{-1} = A^{-1}U$

$V\Sigma^{-1}U^{-1} = A^{-1} \Rightarrow \boxed{A^{-1} = V\Sigma^{-1}U^T}$

(d) Use a combination of your work above to find the singular values for $A^{-1}$?

$\Sigma^{-1} \to \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$ where $\sigma_1$ and $\sigma_2$ are for $A^{-1}$

calculate inverse of $\Sigma \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix}$

$\frac{1}{\det\Sigma}\begin{bmatrix} 5\sqrt{2} & 0 \\ 0 & 10\sqrt{2} \end{bmatrix} = \frac{1}{100}\begin{bmatrix} 5\sqrt{2} & 0 \\ 0 & 10\sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{20} & 0 \\ 0 & \frac{\sqrt{2}}{10} \end{bmatrix}$

$\to \begin{bmatrix} \frac{1}{10\sqrt{2}} & 0 \\ 0 & \frac{1}{5\sqrt{2}} \end{bmatrix}$

$\boxed{\text{singular values of } A^{-1} \text{ in decreasing order: } \sigma_1 = \frac{1}{5\sqrt{2}} \ \& \ \sigma_2 = \frac{1}{10\sqrt{2}}}$