20F-MATH33A-1 Final Exam-Written Portion

[redacted]

TOTAL POINTS

33.5 / 35

QUESTION 1

1 Signature 1 / 1

✓ + **1 pts** Signature Present

QUESTION 2

2 Question 1 6 / 6

✓ + **2 pts** Part (a)
Finding a matrix A.

✓ + **1 pts** Part (b)
Computing the determinant of A.

✓ + **1 pts** Part (b)
Using the determinant of A to find the area of the ellipse.

✓ + **1 pts** Part (c)
Identifying the area of the image as the area of R times the determinant of 2A.

✓ + **1 pts** Part (c)
Computing the determinant of 2A and computing the area of the image.

+ **0 pts** placeholder

QUESTION 3

3 Question 2 8 / 8

✓ + **1 pts** Correctly identified 3 as an eigenvalue

✓ + **1 pts** Correctly described the eigenspace of 3 as a line spanned by the vector u. Must either mention that this is a line or that the geoemetric multiplicity is 1.

✓ + **1 pts** Some explanation or picture as to why 3 is an eigenvalue of A

✓ + **1 pts** Correctly identified $$B=\begin{bmatrix} 3&1\\0&3 \end{bmatrix}$$

✓ + **1 pts** Some explanation for WHY the eigenvalues of A and B are the same, i.e they are similar matrices. Note that A and B are not the same matrix, so if you asserted that A=B then you didn't get this point.

✓ + **1 pts** Some explanation for why 3 is the ONLY eigenvalue of A.

✓ + **1 pts** Correctly identifies the alg multiplicity of 3 as 2.

✓ + **1 pts** Some correct explanation that the matrix A is not diagonalizable.

+ **0.5 pts** Correct geometric multiplicity but incorrect eigenspace, or eigenspace not specified.

+ **0.5 pts** Incorrect or no explanation for why A is not diagonalizable.

+ **0 pts** no credit

QUESTION 4

4 Question 3 7 / 8

✓ + **1 pts** (a) correct answer

✓ + **1 pts** (a) correct justification

✓ + **1 pts** (b) correct answer

✓ + **1 pts** (b) correct justification

✓ + **1 pts** (c) correct answer

✓ + **1 pts** (c) correct justification

✓ + **1 pts** (d) correct answer

+ **1 pts** (d) sufficient justification

QUESTION 5

5 Question 4 6 / 6

✓ + **2 pts** (a) Correct

+ **1 pts**
(a) right, but have missing details or errors in explanation

+ **0.5 pts** I don’t know how to interpret your explanation, or your explanation makes a serious irreparable error

+ **0 pts** (a) incorrect

✓ + **2 pts** (b) Correct

+ **1 pts** (B) Why is your matrix invertible?

+ **1 pts** (B) Correct matrix, but bad explanation: If an nxn matrix has distinct eigenvalues or is symmetric, then it's diagonalizable. But the converse is false. (Need to say something geometric multiplicities or linearly independent eigenvectors, or do the right computation).

+ **1 pts** You said your matrix only has one eigenvector. It has infinitely many. You need to say something about geometric multiplicity or linearly independent vectors, or do the right computation.

+ **1 pts** (b) correct matrix, but incorrect or insufficient explanation

+ **1 pts** (b) incorrect because of small computational error

+ **0 pts** (b) incorrect

✓ + **2 pts** **(c) correct**

+ **1 pts** (C) explained eigenvalues, but not characteristic polynomial

+ **1 pts** (c) correct matrix, but incorrect or insufficient explanation

+ **1 pts** (c) Incorrect because of small computational error

+ **0 pts** (c) incorrect

+ **0 pts** You cited an outside source.

QUESTION 6

6 Question 5 **5.5 / 6**

+ **6 pts** Correct

✓ + **1 pts** **a) computed A^T A**

✓ + **1 pts** **a) correct singular values**

✓ + **1 pts** **b) U correct**

✓ + **0.5 pts** **b) V correct**

+ **0.5 pts** b) SVD correct

✓ + **1 pts** **c) correct**

✓ + **1 pts** **d) correct**

Instructions:

- This exam is in two parts.
  - Part 1 is on CCLE- This is a multiple choice "quiz" that will be graded by the computer, similar to our pre-class assignments.
  - Part 2 is in the following pages and should be written by hand and uploaded to Gradescope.
- You should complete the exam and submit by 8am on Dec 14th PST. Please leave enough time to scan your work into a PDF and upload it to Gradescope! Exams will not be accepted after 8am.
- You may spend as much time as you like on this exam between now and the due date.
- If you do not have a way to print this exam you can copy each question onto a blank piece of paper. BEWARE- Please copy the entire question, and give yourself plenty of space to answer each question. We will take off points if, for example, your answer says

  2) [your answer]

  without including the phrasing of the question 2.
- This exam is open book. You can use any resources you find in our textbook, on our CCLE page or on the internet in general. However you should not have anyone's help to do the exam. So you are not allowed to ask your classmates or TAs about the questions or to post the exam questions on an online forum. Posting our exam questions online is the same asking someone to do the exam for you which is cheating.
- If you have a question about the phrasing of the one of the questions or about the mechanics of completing the exam, you can email ProfRosesMathExamQuestions@gmail.com.

By signing below, I certify that the following exam (including this written part and the part on CCLE) is entirely my own work. I did not receive any help in completing the exam. I will not post the exam questions on public or class forums or discuss the questions with anyone else until after the exam has ended.

(Signature)

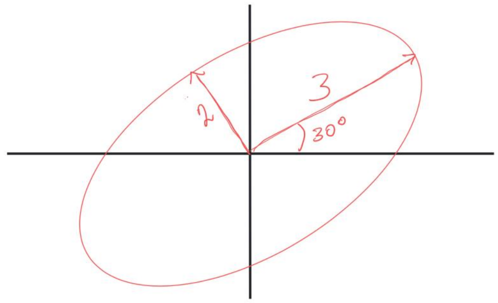
Remember to explain your calculations for full credit! It's fine to use technology to check your answer, but please write out some intermediate steps in your row reductions so we can see your process!

OVER →

1 Signature 1 / 1

✓ + 1 pts **Signature Present**

1. (6 pts) In this class we learned that every 2×2 matrix takes a circle to an ellipse, and also how to find the ellipse using SVD. Now we'll do the reverse. Consider the ellipse pictured below, with semimajor axis of length 3 and at a 30 degree angle with the x-axis, and with and semiminor axis with length 2.



(a) Find a matrix A, that represents a transformation that takes the unit circle to this ellipse. Hint: You may want to think of your transformation as a composition of rotations and scaling.

$$A = \begin{bmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} 3\sqrt{3}/2 & -1 \\ 3/2 & \sqrt{3} \end{bmatrix}}$$

(b) Use the determinant to compute the area of the ellipse.

area of ellipse = det(A)(area of unit circle)

$$= \left[\left(\frac{3\sqrt{3}}{2}\right)(\sqrt{3}) - (-1)\left(\frac{3}{2}\right)\right]\pi = \left(\frac{9}{2} + \frac{3}{2}\right)\pi = \boxed{6\pi}$$

(c) If R is a region of the plane with area equal to 7 units^2, what is the area of the image of this region after it undergoes the transformation represented by the matrix $2A$?

$$2A = 2\begin{bmatrix} 3\sqrt{3}/2 & -1 \\ 3/2 & \sqrt{3} \end{bmatrix} = \begin{bmatrix} 3\sqrt{3} & -2 \\ 3 & 2\sqrt{3} \end{bmatrix}$$

$$\det(2A) = (3\sqrt{3})(2\sqrt{3}) - (-2)(3)$$
$$= 18 + 6$$
$$= 24$$

$$7(\det(2A)) = 7(24) = \boxed{168 \text{ units}^2}$$

OVER →

2 Question 1 **6 / 6**

✓ **+ 2 pts** Part (a)

Finding a matrix A.

✓ **+ 1 pts** Part (b)

Computing the determinant of A.

✓ **+ 1 pts** Part (b)

Using the determinant of A to find the area of the ellipse.

✓ **+ 1 pts** Part (c)

Identifying the area of the image as the area of R times the determinant of 2A.

✓ **+ 1 pts** Part (c)

Computing the determinant of 2A and computing the area of the image.

+ 0 pts placeholder

2. (8 pts) Suppose that the vectors $\bar{u}$ and $\bar{v}$ are a basis $\mathcal{B}$ for $\mathbb{R}^2$.

Suppose further that T is a transformation $\mathbb{R}^2 \to \mathbb{R}^2$ that maps $\bar{u} \mapsto 3\bar{u}$, and $\bar{v} \mapsto \bar{u}+3\bar{v}$. Consider the matrix A such that $T(\bar{x}) = A\bar{x}$.

(a) Arguing geometrically, identify one eigenvalue λ for the matrix A, and describe the eigenspace in terms of $\bar{u}$ and $\bar{v}$. What is the geometric multiplicity of λ?

$A\vec{u} = 3\vec{u}$, therefore $\boxed{\lambda = 3}$.

Since E_3 is spanned only by $\vec{u}$, the image of E_3 is a line, which is a subspace whose dimension = 1.

Therefore, the geometric multiplicity of λ is 1.

(b) Find a matrix B which is the $\mathcal{B}$ matrix for T.

$A\vec{u} = 3\vec{u} = c_1\vec{u} + c_2\vec{v}$ $\quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

$A\vec{v} = \vec{u} + 3\vec{v} = c_1\vec{u} + c_2\vec{v}$ $\quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$[A]_{\mathcal{B}} = B = \boxed{\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}}$

(c) What is the relationship between the eigenvalues of A and the eigenvalues of B? Use your answer to part b, to explain why the matrix A has only one eigenvalue.

A and B share the same eigenvalues (3). B is an upper triangular matrix where both of its diagonal entries are 3. This means B has only one eigenvalue, 3, with algebraic multiplicity of 2. Since B represents the change that A does in respect to basis $\mathcal{B}$, A also has eigenvalue 3 with algebraic multiplicity of 2. According to theorem 7.2.7, an nxn matrix has at most n eigenvalues, counting algebraic multiplicities. Since A is 2x2, that means 3 is the only eigenvalue of A since algmu(3) = 2 and there cannot be any more eigenvalues.

(d) What can you say about the algebraic multiplicity of the eigenvalue λ from part a). Is the matrix A diagonalizable?

The characteristic polynomial of B is $(\lambda-3)^2$. This reveals that there is one eigenvalue λ with algebraic multiplicity 2. A shares the same eigenvalue as B, therefore the algebraic multiplicity of λ from A is also 2.

A is not diagonalizable.

A matrix is diagonalizable if & only if the algebraic multiplicity equals the geometric multiplicity for all eigenvalues.

OVER →

However, the algebraic multiplicity of 3 is 2 while its geometric multiplicity is 1. Since they are unequal, A is not diagonalizable.

3 Question 2 **8 / 8**

✓ **+ 1 pts** **Correctly identified 3 as an eigenvalue**

✓ **+ 1 pts** **Correctly described the eigenspace of 3 as a line spanned by the vector u. Must either mention that this is a line or that the geometric multiplicity is 1.**

✓ **+ 1 pts** **Some explanation or picture as to why 3 is an eigenvalue of A**

✓ **+ 1 pts** **Correctly identified $$B=\begin{bmatrix} 3&1\\0&3 \end{bmatrix}$$**

✓ **+ 1 pts** **Some explanation for WHY the eigenvalues of A and B are the same, i.e they are similar matrices. Note that A and B are not the same matrix, so if you asserted that A=B then you didn't get this point.**

✓ **+ 1 pts** **Some explanation for why 3 is the ONLY eigenvalue of A.**

✓ **+ 1 pts** **Correctly identifies the alg multiplicity of 3 as 2.**

✓ **+ 1 pts** **Some correct explanation that the matrix A is not diagonalizable.**

+ 0.5 pts Correct geometric multiplicity but incorrect eigenspace, or eigenspace not specified.

+ 0.5 pts Incorrect or no explanation for why A is not diagonalizable.

+ 0 pts no credit

3. (8 pts) For each of the following descriptions of a square matrix below, identify it as either an invertible matrix, or not an invertible matrix. Justify your answer using a theorem that we have learned in our class.

(a) A 3×3 matrix whose columns are the vectors $\bar{v}, \bar{u}$ and $2\bar{v} - \bar{u}$

not invertible

According to theorem 3.3.9, invertible matrices must have linearly independent columns. Since $2\vec{v}-\vec{u}$ is a linear combination of its other columns $\vec{v}$ and $\vec{u}$, the columns of A are linearly dependent, causing A to be not invertible.

(b) The following matrix, where a is nonzero

invertible

$$\begin{bmatrix} 0 & a & a \\ a & 0 & a \\ a & a & 0 \end{bmatrix}$$

According to summary 7.1.5, a matrix is invertible if it has nonzero determinants. The determinant of this matrix is $2a^3$, which is nonzero since A is nonzero. Therefore, It is invertible.

(c) A matrix A which is similar to an orthogonal matrix B.

invertible

According to theorem 6.3.1, the determinant of an orthogonal matrix is either 1 or -1.
According to theorem 6.2.7, similar matrices share the same determinant.
Therefore, A will have a determinant of either 1 or -1.
According to summary 7.1.5, matrices are invertible if they have nonzero determinants.
Since A has a nonzero determinant, A is invertible.

(d) A 3×3 matrix representing a transformation T, where T takes a sphere in $\mathbb{R}^3$ to an ellipse in a plane inside $\mathbb{R}^3$.

not invertible

T is a matrix that projects vectors in 3D space to a plane in 3D. Since T transforms vectors that are perpendicular to the plane into the zero vector, this indicates that the kernel of T is nonzero. According to theorem 3.1.7, a square matrix A has $\ker(A) = \{\vec{0}\}$ if and only if A is invertible. Since T is a square matrix whose kernel does NOT equal $\{\vec{0}\}$, T is not invertible.

OVER →

4 Question 3 **7 / 8**

- ✓ **+ 1 pts** (a) correct answer
- ✓ **+ 1 pts** (a) correct justification
- ✓ **+ 1 pts** (b) correct answer
- ✓ **+ 1 pts** (b) correct justification
- ✓ **+ 1 pts** (c) correct answer
- ✓ **+ 1 pts** (c) correct justification
- ✓ **+ 1 pts** (d) correct answer
- **+ 1 pts** (d) sufficient justification

4. (6 pts) Find an example of a matrix with each of the following properties, and explain briefly (1-2 sentences), why the matrix you chose has the desired property. If no such matrix exists, use one of our theorems from class to explain why this can't happen.

(a) A square matrix which represents a transformation that is injective but not surjective.

A matrix is injective if every entry of the transformation gets mapped to a unique value. This can only happen if the rank = # of columns in the matrix, since that would make the kernel $\{\vec{0}\}$ and thus ensure that every output is unique.
A matrix is surjective if for any vector $\vec{v}$, there is a vector $\vec{w}$ so that $A\vec{w} = \vec{v}$. This can only happen if the matrix's rank = # of rows in the matrix.

A square matrix is $n \times n$. If it is injective, then rank = n. If it is surjective, rank = n as well. Therefore, a square matrix can only be injective AND surjective, or neither. It cannot be injective but not surjective.

(b) A matrix that is invertible and has at least one eigenvalue, but is not diagonalizable.

$A = \boxed{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}$

According to theorem 3.3.9, a matrix is invertible if it has independent column vectors.
A has independent column vectors, so it is invertible.

$f_A(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^2$. Therefore, A has one eigenvalue $\lambda = 1$.

$E_1 = \ker \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \text{span} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Geometric multiplicity of $1 = \dim(E_1) = 1$.
Algebraic multiplicity of $1 = 2$. Since geo mult $(1) \neq$ alg mult (1), A is not diagonalizable.

(c) A matrix whose characteristic polynomial is $p(\lambda) = -\lambda(4 - \lambda)(2 - \lambda)^2$

$A = \boxed{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}}$

$p(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 0 & 0 & 0 \\ 0 & 4-\lambda & 0 & 0 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{bmatrix} = -\lambda(4-\lambda)(2-\lambda)^2$

OVER →

A has the desired characteristic polynomial.

5 Question 4 **6 / 6**

✓ **+ 2 pts** **(a) Correct**

+ 1 pts

(a) right, but have missing details or errors in explanation

+ 0.5 pts I don’t know how to interpret your explanation, or your explanation makes a serious irreparable error

+ 0 pts (a) incorrect

✓ **+ 2 pts** **(b) Correct**

+ 1 pts (B) Why is your matrix invertible?

+ 1 pts (B) Correct matrix, but bad explanation: If an nxn matrix has distinct eigenvalues or is symmetric, then it’s diagonalizable. But the converse is false. (Need to say something geometric multiplicities or linearly independent eigenvectors, or do the right computation).

+ 1 pts You said your matrix only has one eigenvector. It has infinitely many. You need to say something about geometric multiplicity or linearly independent vectors, or do the right computation.

+ 1 pts (b) correct matrix, but incorrect or insufficient explanation

+ 1 pts (b) incorrect because of small computational error

+ 0 pts (b) incorrect

✓ **+ 2 pts** **(c) correct**

+ 1 pts (C) explained eigenvalues, but not characteristic polynomial

+ 1 pts (c) correct matrix, but incorrect or insufficient explanation

+ 1 pts (c) Incorrect because of small computational error

+ 0 pts (c) incorrect

+ 0 pts You cited an outside source.

5. (6pts) Consider the matrix

$$A = \begin{bmatrix} 11 & -5 \\ -2 & 10 \end{bmatrix}$$

(a) Compute the singular values for A

$$A^TA = \begin{bmatrix} 11 & -2 \\ -5 & 10 \end{bmatrix}\begin{bmatrix} 11 & -5 \\ -2 & 10 \end{bmatrix} = \begin{bmatrix} 125 & -75 \\ -75 & 125 \end{bmatrix}$$

$f_{A^TA}(\lambda) = \lambda^2 - 250\lambda + 10000 = (\lambda-50)(\lambda-200)$

$\lambda_1 = 200,\ \lambda_2 = 50 \Rightarrow \boxed{\sigma_1 = 10\sqrt{2},\ \sigma_2 = 5\sqrt{2}}$

(b) Compute the SVD for A

$$\boxed{\Sigma = \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix}}$$

$E_{200} = \ker \begin{bmatrix} -75 & -75 \\ -75 & -75 \end{bmatrix} = \text{span}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$E_{50} = \ker \begin{bmatrix} 75 & -75 \\ -75 & 75 \end{bmatrix} = \text{span}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \vec{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\boxed{V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}$$

$\vec{u}_1 = \frac{1}{\sigma_1}A\vec{v}_1 = \frac{1}{10\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)\begin{bmatrix} 16 \\ -12 \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 16 \\ -12 \end{bmatrix} = \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix}$

$\vec{u}_2 = \frac{1}{\sigma_2}A\vec{v}_2 = \frac{1}{5\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)\begin{bmatrix} 6 \\ 8 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$

$$\boxed{U = \begin{bmatrix} 4/5 & 3/5 \\ -3/5 & 4/5 \end{bmatrix}}$$

(c) Given the equation $A = U\Sigma V^T$, prove that, if A is invertible, then $A^{-1} = V\Sigma^{-1}U^T$.

$A^{-1} = (U\Sigma V^T)^{-1} = (V^T)^{-1}\Sigma^{-1}U^{-1}$

$= (V^{-1})^T\Sigma^{-1}U^T = (V^T)^T\Sigma^{-1}U^T$

$= \boxed{V\Sigma^{-1}U^T}$

V is an orthogonal matrix, so $V^{-1} = V^T$

U is also orthogonal, so $U^{-1} = U^T$

(d) Use a combination of your work above to find the singular values for A^{-1}?

Since $A^{-1} = V\Sigma^{-1}U^T$, the singular values of A^{-1} are the diagonal entries of Σ^{-1}.

$$\Sigma^{-1} = \begin{bmatrix} 1/10\sqrt{2} & 0 \\ 0 & 1/5\sqrt{2} \end{bmatrix}$$

therefore the singular values of A^{-1} are $\boxed{\frac{1}{5\sqrt{2}} \text{ and } \frac{1}{10\sqrt{2}}}$

6 Question 5 **5.5 / 6**

+ **6 pts** Correct

✓ + **1 pts** **a) computed A^T A**

✓ + **1 pts** **a) correct singular values**

✓ + **1 pts** **b) U correct**

✓ + **0.5 pts** **b) V correct**

+ **0.5 pts** b) SVD correct

✓ + **1 pts** **c) correct**

✓ + **1 pts** **d) correct**