Math 33A - Lectures 3 and 4
Fall 2018

Midterm 1

Instructions: You have 60 minutes to complete this exam. There are five questions, worth a total of 50 points. This test is closed book and closed notes. No calculator is allowed.

For full credit show all of your work legibly. Unless instructed otherwise, you need to justify your answers. Please write your solutions in the space below the questions; INDICATE if you go over the page and/or use the scrap pages at the end of this booklet.

Please take a moment to ensure that your booklet consists of ten pages, the last three being reserved for additional work.

Do not forget to write your full name, section and UID in the space below. For identification purposes, please sign below.

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Question	Points	Score
1	8	8
2	10	10
3	11	10
4	10	10
5	11	11
Total:	50	49

Problem 1.

For each of the following sentences, give an example of a matrix A with the following properties, or explain why it is impossible.

(a) [4pts.] A is a 3×6 matrix with rank and nullity both equal to 3.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

4

(b) [4pts.] A is a 6×3 matrix with rank and nullity both equal to 3.

$\text{rank}(A) + \text{nullity}(A) = 3 \leftarrow$ # of Columns

4

$3 + 3 \neq 3$ By rank-nullity theorem

Problem 2.

Recall that two vectors $\vec{v}, \vec{w} \in \mathbf{R}^n$ are perpendicular if their dot product is zero: $\vec{v} \cdot \vec{w} = \vec{0}$.

(a) [5pts.] Find a nonzero matrix A such that $A\vec{x}$ is perpendicular to the vector $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ for every $\vec{x} \in \mathbf{R}^3$. [You do not need to justify how you found A, but you do need to show that your choice of A satisfies the prescribed condition.]

$$\vec{x} - \frac{(\vec{x} \cdot \vec{v})}{\vec{v} \cdot \vec{v}} \vec{v} = \vec{x} - \frac{\left(\vec{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

5

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2/3 & -\frac{1}{3} & -1/3 \\ -\frac{1}{3} & 2/3 & -\frac{1}{3} \\ -1/3 & -1/3 & 2/3 \end{bmatrix}$$

$$A\vec{x} \cdot \vec{v} = \tfrac{2}{3}x_1 - \tfrac{1}{3}x_1 - \tfrac{1}{3}x_1 - \tfrac{1}{3}x_2 + \tfrac{2}{3}x_2 - \tfrac{1}{3}x_2 - \tfrac{1}{3}x_3 - \tfrac{1}{3}x_3 + \tfrac{2}{3}x_3 = 0$$

(b) [5pts.] For the matrix A you found in part (a), let $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ be the corresponding linear transformation. Find, with justification, a basis of the image of T.

Basis of $\mathrm{Img}(T)$ is $\left\{ \begin{bmatrix} 2/3 \\ -1/3 \\ -1/3 \end{bmatrix}, \begin{bmatrix} -\frac{1}{3} \\ 2/3 \\ -1/3 \end{bmatrix} \right\}$

5

Because these vectors span $\mathrm{Img}(T)$ and are all linearly independent because

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

two pivot variables
So the corresponding columns form the basis of $\mathrm{Img}(T)$

$$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 3/2 & -3/2 \\ 0 & -3/2 & 3/2 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Problem 3.

Let $\vec{x}, \vec{y}$ be two nonzero vectors in $\mathbf{R}^n$. Consider the set

$$V = \{\vec{v} \in \mathbf{R}^n \text{ such that } \vec{v} \cdot \vec{x} = \vec{v} \cdot \vec{y}\}.$$

(a) [3pts.] Prove that V is a subspace of $\mathbf{R}^n$.

1) $\vec{0} \in V$ because $\vec{0} \cdot \vec{x} = \vec{0} \cdot \vec{y}$ ✓

2) V is closed over multiplication because $k\vec{v} \cdot \vec{x} = k\vec{v} \cdot \vec{y}$ as long as $\vec{v} \cdot \vec{x} = \vec{v} \cdot \vec{y}$ ($\vec{v} \in V$ ✓

3) V is closed over addition because $(\vec{v}+\vec{w}) \cdot \vec{x} = (\vec{v}+\vec{w}) \cdot \vec{y}$
$\vec{v} \cdot \vec{x} + \vec{w} \cdot \vec{x} = \vec{v} \cdot \vec{y} + \vec{w} \cdot \vec{y}$ as long as $\vec{v} \cdot \vec{x} = \vec{v} \cdot \vec{y}$ and $\vec{w} \cdot \vec{x} = \vec{w} \cdot \vec{y}$ $(\vec{v}, \vec{w} \in V)$ ✓

3|3

(b) [3pts.] Let now $\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$ be vectors in $\mathbf{R}^4$ and let V be defined as above.

Show that $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$ belong to V.

3|3

$\vec{v}_1 \rightarrow \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$ $4 = 4$ ✓

$\vec{v}_2 \rightarrow \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$ $1 = 1$ ✓

$\vec{v}_3$ $\begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$ $0 = 0$ ✓

All vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ belong to V by satisfying $\vec{v} \cdot \vec{x} = \vec{v} \cdot \vec{y}$

This problem continues from the previous page. Recall that $\vec{x}, \vec{y}, \vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ are defined in question (b), and $V = \{\vec{v} \in \mathbf{R}^4 \text{ such that } \vec{v} \cdot \vec{x} = \vec{v} \cdot \vec{y}\}$.

(c) [5pts.] Prove that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis of V.

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for V if the vectors span V and are linearly independent.

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \xrightarrow[R_3+R_1,\ R_4-R_1]{R_2-R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

All columns have a leading 1 so they are linearly independent. ✓

$\vec{v} \cdot x = \vec{v} \cdot y \qquad \vec{v} \cdot x - \vec{v} \cdot y = 0 \qquad \vec{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$$2x_1 + 1x_2 + 0x_3 + 1x_4 - 1x_1 - 2x_2 + 1x_3 + 0x_4 = 0$$

$$x_1 - x_2 + x_3 + x_4 = 0$$

$V = \ker \begin{bmatrix} 1 & -1 & 1 & 1 \end{bmatrix}$ since the kernel of this matrix has 3 dimensions, $\dim(V) = 3$

why? Explain

Any 3 linearly independent vectors in V form a basis for V. Since $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in V$, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for V. ✓

4/5

Problem 4.

(a) [5pts.] Using row-reduction find, if it exists, the inverse of the matrix

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 7 & 1 & 0 & 0 \\ 2 & 5 & 8 & 0 & 1 & 0 \\ 3 & 6 & 9 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2-2R_1} \left[\begin{array}{ccc|ccc} 1 & 4 & 7 & 1 & 0 & 0 \\ 0 & -3 & -6 & -2 & 1 & 0 \\ 0 & -6 & -12 & -3 & 0 & 1 \end{array}\right] \xrightarrow[R_3/3]{R_1-R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 3 & -1 & 0 \\ 0 & -3 & -6 & -2 & 1 & 0 \\ 0 & 2 & 4 & 1 & 0 & -\frac{1}{3} \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 3 & -1 & 0 \\ 0 & -1 & -2 & -1 & 1 & -\frac{1}{3} \\ 0 & 2 & 4 & 1 & 0 & -\frac{1}{3} \end{array}\right] \xrightarrow[R_3/2]{R_2\cdot -1} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 3 & -1 & 0 \\ 0 & 1 & 2 & 1 & -1 & \frac{1}{3} \\ 0 & 1 & 2 & \frac{1}{2} & 0 & -\frac{1}{6} \end{array}\right] \xrightarrow{R_3-R_2} \left[\begin{array}{ccc|} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

Inverse not possible because rank(A) = 2 ≠ #colum

(b) [5pts.] Let A be the matrix defined in (a). Find all solutions $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ of the system

$$A\vec{x} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$ [There could be none.]

$$\left[\begin{array}{ccc|c} 1 & 4 & 7 & -1 \\ 2 & 5 & 8 & 0 \\ 3 & 6 & 9 & 1 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 4 & 7 & -1 \\ 0 & -3 & -6 & 2 \\ 0 & -6 & -12 & 4 \end{array}\right] \xrightarrow{R_3/2} \left[\begin{array}{ccc|c} 1 & 4 & 7 & -1 \\ 0 & -3 & -6 & 2 \\ 0 & -3 & -6 & 2 \end{array}\right] \xrightarrow{R_3-R_2} \left[\begin{array}{ccc|c} 1 & 4 & 7 & -1 \\ 0 & -3 & -6 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 7 & -1 \\ 0 & 1 & 2 & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1-4R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1+4(\frac{2}{3}) \\ 0 & 1 & 2 & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad -\frac{3}{3}+\frac{8}{3} = \frac{5}{3}$$

$$\boxed{\vec{X} = \begin{bmatrix} \frac{5}{3}+z \\ -\frac{2}{3}-2z \\ z \end{bmatrix}} \quad \forall z \in \mathbb{R}$$

infinite solutions.

Problem 5.

Let P be the plane in $\mathbf{R}^3$ given by the equation $x - y + z = 0$. $\quad \vec{n} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

(a) [5pts.] Let $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ be the reflection across the plane P. Find the matrix of T with respect to the standard basis of $\mathbf{R}^3$.

$$\vec{V} = \vec{V}^{\parallel \vec{n}} + V^{\perp \vec{n}} \qquad T(\vec{V}) = V^{\perp \vec{n}} - V^{\parallel \vec{n}} + \left(V^{\parallel \vec{n}} - V^{\parallel \vec{n}}\right)$$

$$= \vec{V} - 2 v^{\parallel \vec{n}}$$

$$= \vec{V} - 2\left(\frac{\vec{V} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}{3}\right)\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix} \vec{x}$$

(b) [6pts.] Find a basis $\mathfrak{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of $\mathbf{R}^3$ such that the $\mathfrak{B}$-matrix of T is diagonal. Write down the $\mathfrak{B}$-matrix of T.

$$\mathcal{B} = \Big\{ \underbrace{\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}_{\vec{V}_1}, \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{V}_2}, \underbrace{\frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}}_{\vec{V}_3} \Big\}$$

$$T(\vec{v}_1) = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{B}} \qquad [T(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [\vec{x}]_{\mathcal{B}}$$

$$T(\vec{V}_2) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{B}}$$

$$T(\vec{V}_3) = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{\mathcal{B}}$$