33A/1 Linear Algebra and Applications: Midterm Exam 2

Name: Andrew Lee

UID: 804621520

Question 1

Let $A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$.

(a) *[2 points]* Find all eigenvalues of A and their algebraic multiplicities.

(b) *[3 points]* Find the eigenspaces and geometric multiplicities of the eigenvalues.

(c) *[2 points]* Is A diagonalizable? Justify your answer.

(d) *[3 points]* Two of the following four statements are true. Identify these and give a brief justification:

- $E_0 = \ker(A)$
- ~~$E_0 \subseteq E_1$~~
- $E_1 \subseteq \text{im}(A)$
- $E_1 = \text{im}(A)$,

where E_{λ_i} denotes the eigenspace of an eigenvalue λ_i of A.

....

Triangular matrix

a. $\lambda_1 = 0 \qquad \lambda_2 = 1$

2

$\text{alg}(0) = 0 \qquad \text{alg}(1) = 2$

b. $A\vec{v} = \lambda\vec{v}$

3

$\lambda = 0 \quad A\vec{v} = 0$ Since $\vec{v} \neq \vec{0}$, find $\ker(A)$

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 + x_2 = 0 \quad x_1 = -x_2$

$x_3 = 0$

$x_2 = t$

$\ker(A) = \begin{pmatrix} -t \\ t \\ 0 \end{pmatrix} = \vec{v}$

$$\boxed{E_0 = \text{span}(\vec{v}) = \text{span}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \text{geo}(0) = \dim(E_0) = 1}$$

$\lambda = 1$

$A\vec{v} = 1\vec{v}$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$0 = v_1$

$\cancel{v_1} + v_2 = v_2$

$\cancel{v_1} + v_2 + v_3 = v_3$

$v_2 = 0$

$v_1 = 0$

$v_2 = 0$

$v_3 = v_3$ ← free variable

$$\boxed{E_1 = \text{span}\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) \quad \text{geo}(1) = 1}$$

c. no, A is not diagonalizable because $geo(\lambda_1) + geo(\lambda_2) \neq n$,

2 means the eigenvectors do not form an eigenbasis of A, so it is not diagonalizable

d. $E_0 = \ker(A)$ is true.

3 E_0 is the span of the eigenvectors whose associated eigenvalue is 0.

$A\vec{v} = \lambda\vec{v}$

$A\vec{v} = \vec{0}$

$\vec{v} \in \ker(A)$

$\text{span}(\vec{v}) = E_0$

— $E_1 \subseteq \text{im}(A)$. $\text{im}(A) = \text{span}\left(\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right)$

$E_1 = \text{span}\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right) \subseteq \text{span}\left(\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right)$

Question 2

Consider the projection in $\mathbb{R}^2$ to the line spanned by $\begin{pmatrix}4\\0\end{pmatrix}$.

(a) *[2 points]* Use the vector above to show that the projection matrix A for this transformation is $\begin{pmatrix}1&0\\0&0\end{pmatrix}$.

(b) *[2 points]* Consider the following basis: $\mathfrak{B} = \left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right)$. Find $\begin{pmatrix}5\\-3\end{pmatrix}_{\mathfrak{B}}$.

(c) *[1 points]* Give a short geometric interpretation of the change of basis in (b).

(d) *[3 points]* Find the $\mathfrak{B}$-matrix B of A, and explain the resulting matrix geometrically.

(e) *[2 points]* Consider other bases $\mathfrak{B}' = \left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix}\right)$ and $\mathfrak{B}'' = \left(\begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right)$. Without explicit computation, explain why $A = B' = B''$, where B' and B'' are the $\mathfrak{B}'$-matrix and $\mathfrak{B}''$-matrix of A, respectively.

$\vec{v}\cdot\vec{x} = \vec{v}^T\vec{x}$

………

a. $\vec{v} = \begin{pmatrix}4\\0\end{pmatrix}$ $\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 4,0\rangle}{4} = \langle 1,0\rangle$ $\vec{u} = \begin{pmatrix}u_1\\u_2\end{pmatrix}$

$\text{proj}_{\vec{v}}\vec{x} = \begin{pmatrix}u_1^2 & u_1u_2\\ u_1u_2 & u_2^2\end{pmatrix} = \begin{pmatrix}1&0\\0&0\end{pmatrix}$ ✓ 2

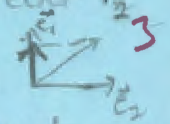
b. $S\vec{x} = \begin{pmatrix}5\\-3\end{pmatrix}$ $[\vec{x}]_{\mathfrak{B}} = S^{-1}\vec{x} = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}5\\-3\end{pmatrix} = \boxed{\begin{pmatrix}-3\\5\end{pmatrix}}$ 2

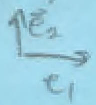
c. switch the values of x and y; reflect about line x=y 1

d. $AS = SB$ $B = S^{-1}AS = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}$

$\begin{array}{c}B = \begin{pmatrix}0&0\\0&1\end{pmatrix}\\ \quad\ \ \vec{v}_1\ \ \vec{v}_2\end{array}$ projection onto the 2nd basis vector $\vec{v}_2$ with respect to basis $\left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right)$ 3

$\vec{v}_1 = \vec{e}_2$ $\vec{v}_2 = \vec{e}_1$



e. $\mathfrak{B}$, $\mathfrak{B}'$, and $\mathfrak{B}''$ all transform the standard bases  into a form of  meaning that B' and B'' will be similar to ~~the~~ A. → just rotated in different forms ✓

$A = SBS^{-1}$

$\swarrow$

$A = (\mathcal{B}')\,\mathcal{B}'\,(\mathcal{B}')^{-1}$

$\longrightarrow A = (\mathcal{B}'')\,\mathcal{B}''\,(\mathcal{B}'')^{-1}$

this would not be true if

$\mathcal{B}''' = \left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 1/\sqrt{2} \end{pmatrix} \right)$

10

Question 3

(a) *[2 points]* Is the following set of vectors orthonormal?

$$\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(b) *[3 points]* Let $V = \text{span}\left(\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$. Let $\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Find the decomposition of $\vec{x}$ into $\vec{x}^{\parallel} = proj_V\vec{x}$ and $\vec{x}^{\perp} = proj_{V^\perp}\vec{x}$.

(c) *[3 points]* Consider the following data of three students and the number of pets they have owned over their lifetimes.

Student	**# Cats**	**# Dogs**
A	4	1
B	5	0
C	0	5

Find the correlation coefficient between the number of cats and number of dogs students in the sample have owned.

(d) *[2 points]* Draw a picture that illustrates the correlation coefficient of this particular example geometrically.

....

a. orthonormal: $\vec{v_1} \cdot \vec{v_2} = \vec{v_1} \cdot \vec{v_3} = \vec{v_2} \cdot \vec{v_3} = 0$ ✓

✓

$|\vec{v_1}| = |\vec{v_2}| = |\vec{v_3}| = 1$ ✓

Yes

b. $proj_V \vec{x} = (\vec{x} \cdot \vec{u_1})\vec{u_1} + (\vec{x} \cdot \vec{u_2})\vec{u_2}$ $\quad \vec{u_1} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$ $\vec{u_2} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$= \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} + 0\right)\vec{u_1} + (0 + 0 + 1)\vec{u_2}$

✓

$$\boxed{proj_V \vec{x} = \vec{u_2}}$$

$proj_{V^\perp}\vec{x} = \vec{x}^{\perp} = \vec{x} - proj_V \vec{x}$

$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \vec{x}^{\perp}}$

c. $\vec{v} = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix}$ $\vec{w} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$. Normalize.

$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ $\frac{v_1 + v_2 + v_3}{3} = 3$

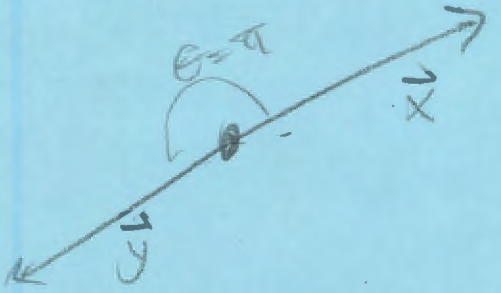
$\vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ $\frac{w_1 + w_2 + w_3}{3} = 2$

$\vec{x} \Rightarrow \begin{pmatrix} 4-3 \\ 5-3 \\ 0-3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1-2 \\ 0-2 \\ 5-2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix} = \vec{y}$

$$r = \frac{\vec{x} \cdot \vec{y}}{|\vec{x}||\vec{y}|} = \frac{-1-4-9}{\sqrt{1+4+9}\sqrt{1+4+9}} = \frac{-(\cancel{1+4+9})}{(\cancel{1+4+9})} = \boxed{-1 = r}$$

d. $r = -1$ means $\cos\theta = \pi$, meaning the [illegible] normalized vectors $\vec{x}$ and $\vec{y}$ are in opposite directions, meaning they are inversely correlated.



Q1	Q2	Q3	TOTAL
10	8	10	~~2~~ 28