Question 1

Consider the coefficient matrix

$$A = \begin{pmatrix} 3 & -1 & 2 \\ -1 & 1 & 0 \\ 2 & -2 & -1 \end{pmatrix}.$$

(a) *[2 points]* Give the definition of the rank of a matrix, *rank*(A).

(b) *[3 points]* Write down the system of equations and the augmented matrix that represent

$$A\vec{x} = \begin{pmatrix} 10 \\ -2 \\ 1 \end{pmatrix}.$$

(c) *[3 points]* Write the augmented matrix from **(b)** in reduced row echelon form, and use this to find a solution for $\vec{x}$ that satisfies the system of equations from **(b)**.

(d) *[2 points]* Is the solution for $\vec{x}$ that you found in **(c)** the only solution for the system of equations from **(b)**? Justify your answer.

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(a) The rank of a matrix, rank(A), means the dimension of the image of A. It can also be interpreted as the number of rows in rref(A) that have leading 1's.

(b)
$$A\vec{x} = \begin{bmatrix} 3 & -1 & 2 \\ -1 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} + x_3\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 10 \\ -2 \\ 1 \end{bmatrix}$$

or

$$\boxed{\begin{cases} 3x_1 - x_2 + 2x_3 = 10 \\ -x_1 + x_2 = -2 \\ 2x_1 - 2x_2 - x_3 = 1 \end{cases}}$$

Augmented matrix:
$$\boxed{\left[\begin{array}{ccc|c} 3 & -1 & 2 & 10 \\ -1 & 1 & 0 & -2 \\ 2 & -2 & -1 & 1 \end{array}\right]}$$

(c)
$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 10 \\ -1 & 1 & 0 & -2 \\ 2 & -2 & -1 & 1 \end{array}\right] \xrightarrow{\text{swap rows}} \left[\begin{array}{ccc|c} -1 & 1 & 0 & -2 \\ 3 & -1 & 2 & 10 \\ 2 & -2 & -1 & 1 \end{array}\right] \times(-1) \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 3 & -1 & 2 & 10 \\ 2 & -2 & -1 & 1 \end{array}\right] \begin{array}{l} \\ -3R_1 \\ -2R_1 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & +1 & +3 \end{array}\right] \div 2 \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} +R_2 \\ -R_3 \\ \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] -R_3 \longrightarrow \boxed{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]} \Rightarrow \boxed{\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}}$$

(d) Yes, it is the only solution. The rref of the coefficient matrix is an identity matrix, which means there is only one solution.

10

Question 2

Consider the following two linear transformations, $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $L : \mathbb{R}^2 \to \mathbb{R}^2$:

$$\vec{y} = T(\vec{x}) = A\vec{x} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \vec{x},$$

$$\vec{z} = L(\vec{y}) = B\vec{y} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{y},$$

$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \to \theta = \frac{\pi}{2} = 90^\circ$

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$

where $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^2$.

(a) *[4 points]* Explain in words, precisely, what the effects of the linear transformations T and L are. Use words such as rotation, angle, scaling, projection, reflection, shear, *etc.*

(b) *[4 points]* Find the matrix of the composite linear transformation $\vec{z} = L(T(\vec{x}))$, and explain its effect.

(c) *[2 points]* Find the inverse of the composite function above, namely, find the matrix for $\vec{x} = T^{-1}(L^{-1}(\vec{z}))$.

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(a) T has the effect of scaling by 2. It makes the magnitude of a vector (or length) twice as long.
L has the effect of rotating by 90%. counterclockwise.

(b) $L(T(\vec{x})) = LT(\vec{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}(\vec{x}) = \begin{bmatrix} 0\cdot 2 - 1\cdot 0 & 0\cdot 0 - 1\cdot 2 \\ 1\cdot 2 + 0\cdot 0 & 1\cdot 0 + 0\cdot 2 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}(\vec{x})}$

$L(T(\vec{x}))$ has the effect of first scaling by 2 and then rotating by 90%. counterclockwise.

(c) $\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \Rightarrow \det = ad - bc = 0 - (-4) = 4.$

$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\frac{1}{4}\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}$

verify:

$\left[\begin{array}{cc|cc} 0 & -2 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 2 & 0 & 0 & 1 \\ 0 & -2 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & 0 \end{array}\right]$

$\therefore \boxed{\begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}}$

Question 3

(a) *[3 points]* Define the kernel and image of a matrix A.

(b) *[4 points]* Verify that both the kernel and image of A are closed under addition and scalar multiplication.

(c) *[3 points]* Consider an $n \times p$ matrix A and a $p \times m$ matrix B. Show that $\ker(B) \subseteq \ker(AB)$.

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10/10

(a) The kernel of a matrix A is $\{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$. This means that the kernel is a set of vectors that map to $\vec{0}$ under the transformation whose matrix is A.

3

The image of a matrix A is $\{\vec{y} \in \mathbb{R}^n \mid A\vec{x} = \vec{y}\}$. This means that the image is a set of all possible vectors that can be generated as a result of the transformation whose matrix is A.

(b) I. Consider 2 arbitrary vectors in $\ker(A)$, $\vec{a}$ and $\vec{b}$.

This means $A\vec{a} = A\vec{b} = \vec{0}$.

Then, $A(\vec{a}+\vec{b}) = A(\vec{a}) + A(\vec{b}) = \vec{0}+\vec{0} = \vec{0}$. (because linear transformation)

$$A(k\vec{a}) = A(\underbrace{\vec{a}+\vec{a}+\cdots+\vec{a}}_{k\text{ times}}) = \underbrace{A(\vec{a})+A(\vec{a})+\cdots+A(\vec{a})}_{k\text{ times}} = \underbrace{\vec{0}+\vec{0}+\cdots+\vec{0}}_{k\text{ times}} = \vec{0}$$

where $k \in \mathbb{R}$

$\therefore$ $\ker(A)$ is closed under addition and scalar multiplication ■

II. Consider 2 arbitrary vectors in $\operatorname{im}(A)$, $\vec{c}$ and $\vec{d}$.

4

This means there exist $\vec{x}_1$ and $\vec{x}_2$ such that $A\vec{x}_1 = \vec{c}$ and $A\vec{x}_2 = \vec{d}$.

Then, $\vec{c}+\vec{d} = A\vec{x}_1 + A\vec{x}_2 = A(\vec{x}_1+\vec{x}_2)$. $\Rightarrow$ So $\vec{c}+\vec{d}$ is also in $\operatorname{im}(A)$.

$$k\vec{c} = \underbrace{\vec{c}+\vec{c}+\cdots+\vec{c}}_{k\text{ times}} = \underbrace{A\vec{x}_1+A\vec{x}_1+\cdots+A\vec{x}_1}_{k\text{ times}} = A(\underbrace{\vec{x}_1+\vec{x}_1+\cdots+\vec{x}_1}_{k\text{ times}}) = A(k\vec{x}_1).$$ $\Rightarrow$ so $k\vec{c}$ is still in im

$\therefore$ $\operatorname{im}(A)$ is closed under addition and scalar multiplication ■

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(c) If $B\vec{x} = \vec{0}$, then $A(B\vec{x}) = A(\vec{0}) = \vec{0}$, but $A(B\vec{x}) = AB(\vec{x})$.

This means that every $\vec{x} \in \ker(B)$ is also an element of $\ker(AB)$

Therefore, $\ker(B) \subseteq \ker(AB)$

3