33A/1 Linear Algebra and Applications: Midterm Exam 1

10/23/2015

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Question 1

A subspace V of $\mathbb{R}^n$ is called a *hyperplane* if the vectors $\vec{x} \in V$ are defined by an equation: $a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$, where at least one of the coefficients a_i is nonzero. lin. dependent

(a) *[3 points]* How many of the variables x_i are free? What is the dimension of a hyperplane in $\mathbb{R}^n$?

(b) *[4 points]* Explain what a hyperplane in $\mathbb{R}^2$ looks like, and give a basis for the hyperplane in $\mathbb{R}^2$ given by the equation $x_1 + 3x_2 = 0$.

(c) *[3 points]* Find an equation like the one given above for the plane spanned by the vectors $\begin{pmatrix}1\\0\\1\end{pmatrix}$ and $\begin{pmatrix}1\\0\\0\end{pmatrix}$ in $\mathbb{R}^3$.

....

why?

a. at most $n-1$ variables are free. /

Dimension: ~~at~~ most $n-1$

why?

not vectors!

b. $a_1\vec{x}_1 + a_2\vec{x}_2 = \vec{0}$ Let $a_1 \neq 0$

Then $\vec{x}_1$ and $\vec{x}_2$ are not linearly independent

then $\vec{x}_1 = c\vec{x}_2$

then a hyperplane in $\mathbb{R}^2$ is a line

this answer is accidentally correct. your reasoning is not correct.

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$x_1 + 3x_2 = 0$

$(1\ 3 \mid 0)$

free variable $x_2 = t$

$x_1 + 3t = 0$

$x_1 = -3t$

$x_2 = t$

$\boxed{\begin{pmatrix}-3t\\t\end{pmatrix} \ t \in \mathbb{R}}$

c. $\operatorname{span}\left(\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}\right) = a_1\begin{pmatrix}1\\0\\1\end{pmatrix} + a_2\begin{pmatrix}1\\0\\0\end{pmatrix}$ $a_1, a_2 \in \mathbb{R}$

$\left(\begin{array}{cc|c}1&1&0\\0&0&0\\1&0&0\end{array}\right) \rightarrow \left(\begin{array}{cc|c}1&1&0\\0&0&0\\0&-1&0\end{array}\right)$

$\boxed{x_2 = t \ t \in \mathbb{R}}$

$\begin{array}{cc|c}a_1&a_2&\\ \hline 1&0&0\\0&0&0\\0&1&0\end{array}$

$a_1 = 0$

$a_2 = 0$

not right form of equation

Question 2

(a) *[3 points]* Define the kernel and image of a matrix A.

(b) *[4 points]* Verify that both the kernel and image of A are closed under addition and scalar multiplication.

on last page → **(c)** *[3 points]* Find a 3×3 matrix A such that both the kernel and the image of A contain $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$.

....

a. ker(A) is the set of vectors $\vec{x}$ such that $A\vec{x} = \vec{0}$

im (A) is the set of vectors $\vec{y}$ such that $A\vec{x} = \vec{y}$ for ~~any~~ some $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$

such that $x_1, x_2, \dots x_m \in \mathbb{R}$

b. Let $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$ be in ker(A).

Kernel:

→ Closed under addition:

unnecessarily long

$A\vec{a} = \vec{0}$ and $A\vec{b} = \vec{0}$ $\quad A = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_m \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}$

$A\vec{a} + A\vec{b} = \vec{0}$

$$\begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_m \\ \downarrow & \downarrow & & \downarrow \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} + \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_m \\ \downarrow & \downarrow & & \downarrow \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} = \vec{0}$$

$$(a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_m\vec{v}_m) + (b_1\vec{v}_1 + b_2\vec{v}_2 + \dots + b_m\vec{v}_m) = \vec{0}$$

$$(a_1+b_1)\vec{v}_1 + (a_2+b_2)\vec{v}_2 + \dots + (a_m+b_m)\vec{v}_m = \vec{0}$$

$$\begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ v_1 & v_2 & \dots & v_m \\ \downarrow & \downarrow & & \downarrow \end{pmatrix} \begin{pmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_m+b_m \end{pmatrix} = \vec{0}$$

$$A \begin{pmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_m+b_m \end{pmatrix} = \vec{0}$$

Thus, $\vec{a}+\vec{b}$ is also in ker(A).

Thus, ker(A) is closed under addition

Kernel:

→ Closed under scalar multiplication:

Let $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$ be in $\ker(A)$

$A\vec{a} = \vec{0}$. Let $c \in \mathbb{R}$ - constant

$cA\vec{a} = c\vec{0}$

$c \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ \downarrow & \downarrow & & \downarrow \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} = \vec{0}$

$c\left(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_m\vec{v}_m\right) = \vec{0}$

$ca_1\vec{v}_1 + ca_2\vec{v}_2 + \cdots + ca_m\vec{v}_m = \vec{0}$

$A \begin{pmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_m \end{pmatrix} = \vec{0}$. Thus, $\vec{b} = \begin{pmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_m \end{pmatrix}$ is also in $\ker(A)$

Thus, $\ker(A)$ is closed under scalar multiplication.

Image:

→ Addition: Let $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$ be in $\mathrm{im}(A)$

$A\vec{x}_1 = \vec{a}$ and $A\vec{x}_2 = \vec{b}$

$A\vec{x}_1 + A\vec{x}_2 = \vec{a} + \vec{b}$

$A(\vec{x}_1 + \vec{x}_2) = \vec{a} + \vec{b}$

Since $\vec{x}_1$ and $\vec{x}_2$ have components that can be any real number, $\vec{x}_1 + \vec{x}_2$ has components that can be any real number, so

$\vec{x}_3 = \vec{x}_1 + \vec{x}_2$

$A\vec{x}_3 = \vec{a} + \vec{b}$. Thus, $\vec{a} + \vec{b}$ is in $\mathrm{im}(A)$

Thus, $\mathrm{im}(A)$ is closed under addition.

→ Scalar multiplication:

$A\vec{x}_a = \vec{a}$

$cA\vec{x}_a = c\vec{a}$

$\vec{x}_a = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$

$A \begin{pmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_m \end{pmatrix} = c\vec{a}$

$A(\vec{cx}) = c\vec{a}$

$A\vec{x}_b = c\vec{a}$

Thus, $\mathrm{im}(A)$ is closed under scalar multiplication

2c. $\vec{v} \in \ker(A)$ $\vec{v} \in \operatorname{im}(A)$ $\vec{v} = \begin{pmatrix}1\\2\\4\end{pmatrix}$

$$A = \begin{pmatrix}1 & 0 & -1/4\\ 2 & 0 & -1/2\\ 4 & 0 & -1\end{pmatrix}\begin{pmatrix}1\\2\\4\end{pmatrix} = \vec{0} \qquad A\vec{x} = \begin{pmatrix}1\\2\\4\end{pmatrix}$$

and $A\begin{pmatrix}1\\2\\4\end{pmatrix} = \vec{0}$

$$\boxed{A = \begin{pmatrix}1 & 0 & -1/4\\ 2 & 0 & -1/2\\ 4 & 0 & -1\end{pmatrix}}$$

Question 3

(a) *[3 points]* Define what it means for a matrix to be invertible.

(b) *[4 points]* Find the projection of the vector $\vec{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ onto the line L spanned by $\vec{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Explain why this transformation is not invertible.

(c) *[3 points]* For which values of a is the following matrix invertible? First answer this question using the determinant of A, and then give a geometrical interpretation.

$$A = \begin{pmatrix} a & 1 \\ -1 & a \end{pmatrix}$$

....

a. A matrix is invertible if there exists a matrix A^{-1} such that $A^{-1}(\vec{b}) = \vec{a}$, where $A(\vec{a}) = \vec{b}$. This means for every $\vec{x}$ there is only one solution $\vec{y}$, and for every $\vec{y}$ there is only one solution $\vec{x}$. ~~It also means~~

b. $\text{proj}_L \vec{x} = \left(\frac{\vec{x} \cdot \vec{w}}{\|w\|^2}\right) \vec{w}$ $\quad \|\vec{w}\| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\vec{x} \cdot \vec{w} = 0 + 1 = 1$

$= \frac{1}{2} \vec{w} = \frac{1}{2} \langle 1, 1 \rangle = \boxed{\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}}$

This is not invertible because the projection of ~~[illegible]~~ $\vec{x}$ onto L causes it to lose information

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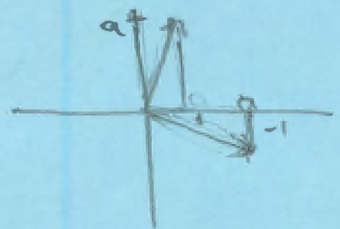
why? how? that's not a very precise statement.

c. not invertible iff $\det(A) = 0$

$\det(A) = a^2 - (-1)(1) = a^2 + 1 = 0$

$a^2 = -1 \quad a = i$ or $a = -i$

<u>If $a \in \mathbb{R}$, then A is invertible</u>



The column vectors $\begin{pmatrix} a \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ a \end{pmatrix}$ are $\perp$ to each other, no matter what real value a is. Thus, if $a \in \mathbb{R}$, A is invertible

Scratch: $\vec{a} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\vec{n} = \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = (0-0)\vec{i} - (0-1)\vec{j} + (0)\vec{k}$$

$$\langle 0, 1, 0 \rangle$$

$$\vec{n} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Q1	Q2	Q3	TOTAL
5	10	8	23