This exam is DUE Wednesday, June 10th at 8AM Pacific Time on CCLE.

You have 24 hours to complete and submit this exam, from 8:00 AM Pacific Time on Tuesday, June 9th, to 8:00 AM Pacific Time on Wednesday, June 10th. I have designed this exam to have the same length as if you only had 3 hours to complete it; it is your responsibility to save time for uploading. If you encounter technical difficulties, please email me immediately.

If you have any questions, email me at rhousden@math.ucla.edu

Good luck!

Your Name: ____________________

Your Student ID: ____________________

1. (10 points) Indicate which of the following are true or false; no justification is required:
   (a) (1 point) Every 2×2 reflection matrix is positive definite. **(T/F)**
   (b) (1 point) 0-dimensional subspaces do not exist. **(T/F)**
   (c) (1 point) The equation $(A+B)(A-B) = A^2 - AB + BA - B^2$ is true for all 2×2 matrices A and B. **(T/F)**
   (d) (1 point) Every diagonalizable matrix with real coefficients is symmetric. **(T/F)**
   (e) (1 point) Every 2×2 shearing matrix is diagonalizable over $\mathbb{R}$. **(T/F)**
   (f) (1 point) The list $(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix})$ is linearly independent. **(T/F)**
   (g) (1 point) The equation $det(A) = det(rref(A))$ is true for all 5×5 matrices A. **(T/F)**
   (h) (1 point) Every 2×2 matrix with 2 distinct eigenvalues is diagonalizable. **(T/F)**
   (i) (1 point) The equation $(A^T)^{-1} = (A^T)(A^{-1})$ is true for every invertible 3×3 matrix A. **(T/F)**
   (j) (1 point) Every 2×2 reflection matrix is orthogonal. **(T/F)**

$A^2 - AB + BA - B^2$

2. (10 points) Let T be the linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with matrix

$$\mathrm{A} = \begin{bmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 5 \end{bmatrix}$$

(a) (2 points) Without doing any computations, we know A is diagonalizable. Why?

A is symmetric and all symmetric matrices are (orthogonally) diagonalizable by the Spectral Theorem.

(b) (4 points) Find the eigenvalues of A. (You must show your steps to receive credit.)

$$f_A(\lambda) = \det \begin{bmatrix} 3-\lambda & -\sqrt{3} \\ -\sqrt{3} & 5-\lambda \end{bmatrix}$$
$$= (3-\lambda)(5-\lambda) - 3$$
$$= 15 - 3\lambda - 5\lambda + \lambda^2 - 3$$
$$= \lambda^2 - 8\lambda + 12$$
$$= (\lambda - 6)(\lambda - 2)$$

$\lambda = 6, 2$

(c) (4 points) Find an eigenbasis of A. (You must show your steps to receive credit.)

$$E_6 = \ker(A - 6I_2) = \ker \begin{bmatrix} -3 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix} \right)$$

$$E_2 = \ker(A - 2I_2) = \ker \begin{bmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} \right)$$

$$\left(\begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}, \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} \right)$$

3. (10 points) Consider the quadratic form:

$$q(x_1, x_2, x_3) = 6x_1x_2 + x_3^2$$

(a) (4 points) Write the (symmetric) matrix, A, of this quadratic form.

$$A = \begin{bmatrix} 0 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) (4 points) Find the eigenvalues of A. (You must show your steps to receive credit.)

$$f_A(\lambda) = \det \begin{bmatrix} -\lambda & 3 & 0 \\ 3 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

$$= (-\lambda)^2(1-\lambda) - 9(1-\lambda)$$

$$= (1-\lambda)(\lambda^2 - 9)$$

$$= (1-\lambda)(\lambda+3)(\lambda-3)$$

$$\lambda = 1, \pm 3$$

(c) (2 points) Which of the following adjectives describe A? Circle all that apply.

- **Positive Definite**
- **Positive Semidefinite**
- **Indefinite**
- **Negative Semidefinite**
- **Negative Definite**

4. (10 points) Find the curve of the form $f(x) = c_0 + c_1 x + c_2 x^2$ that best fits the points (-2,-1), (0,-1), (1,1), and (1,3). (You must show your steps to receive credit.)

$$c_0 + c_1(-2) + c_2(4) = -1$$
$$c_0 + c_1(0) + c_2(0) = -1$$
$$c_0 + c_1(1) + c_2(1) = 1$$
$$c_0 + c_1(1) + c_2(1) = 3$$

$$\underbrace{\begin{bmatrix} 1 & -2 & 4 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} -1 \\ -1 \\ 1 \\ 3 \end{bmatrix}}_{\vec{b}}$$

$$A^T A \vec{x} = A^T \vec{b}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & 0 & 1 & 1 \\ 4 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & 0 & 1 & 1 \\ 4 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1+1+1+1 & -2+1+1 & 4+1+1 \\ -2+1+1 & 4+1+1 & -8+1+1 \\ 4+1+1 & -8+1+1 & 16+1+1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1-1+1+3 \\ 2+1+3 \\ -4+1+3 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 & 6 \\ 0 & 6 & -6 \\ 6 & -6 & 18 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 4 & 0 & 6 & 2 \\ 0 & 6 & -6 & 6 \\ 6 & -6 & 18 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R1,\ \frac{1}{6}R2,\ \frac{1}{3}R3} \left[\begin{array}{ccc|c} 2 & 0 & 3 & 1 \\ 0 & 1 & -1 & 1 \\ 2 & -2 & 6 & 0 \end{array}\right]$$

$$\xrightarrow{R3-R1} \left[\begin{array}{ccc|c} 2 & 0 & 3 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & -2 & 3 & -1 \end{array}\right] \xrightarrow{R3+2R2} \left[\begin{array}{ccc|c} 2 & 0 & 3 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R2+R3} \left[\begin{array}{ccc|c} 2 & 0 & 3 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R1-3R3} \left[\begin{array}{ccc|c} 2 & 0 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$f(x) = -1 + 2x + x^2$$

5. (10 points) Find a Singular Value Decomposition for:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

You must show your steps to receive credit.

$$A^TA = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4+1 & 2+2 \\ 2+2 & 1+4 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

$$f_{A^TA}(\lambda) = \det\begin{bmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{bmatrix}$$
$$= (5-\lambda)^2 - 16$$
$$= 25 - 10\lambda + \lambda^2 - 16$$
$$= \lambda^2 - 10\lambda + 9$$
$$= (\lambda-9)(\lambda-1)$$

$\lambda = 9, 1$

$\sigma_1^2 = 9 \quad \sigma_1 = 3$

$\sigma_2^2 = 1 \quad \sigma_2 = 1$

$$\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E_9 = \ker(A^TA - 9I_2) = \ker\begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

$$E_1 = \ker(A^TA - I_2) = \ker\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} = \text{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$$

$$\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \qquad V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\vec{u}_1 = \frac{1}{\sigma_1}A\vec{v}_1 = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{3\sqrt{2}}\begin{bmatrix} 2+1 \\ 1+2 \end{bmatrix} = \frac{1}{3\sqrt{2}}\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sigma_2}A\vec{v}_2 = \frac{1}{1}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 2-1 \\ 1-2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A = U\Sigma V^T$$
$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

6. (10 points) Find all solutions to the system:

$$\begin{array}{rcrcrcl}
\frac{1}{\sqrt{3}}x_1 & + & \frac{1}{\sqrt{3}}x_2 & + & \frac{1}{\sqrt{3}}x_3 & = & \sqrt{3} \\
\frac{1}{\sqrt{6}}x_1 & + & (-\frac{2}{\sqrt{6}})x_2 & + & \frac{1}{\sqrt{6}}x_3 & = & \sqrt{6} \\
\frac{1}{\sqrt{2}}x_1 & + & 0x_2 & + & (-\frac{1}{\sqrt{2}})x_3 & = & \sqrt{2}
\end{array}$$

You must show your steps to receive credit. If there are no solutions, write "No Solutions."

$$\underbrace{\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} \sqrt{3} \\ \sqrt{6} \\ \sqrt{2} \end{bmatrix}}_{\vec{b}}$$

$$\left\| \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{6} \\ 1/\sqrt{2} \end{bmatrix} \right\| = \sqrt{\frac{1}{3}+\frac{1}{6}+\frac{1}{2}} = \sqrt{\frac{2+1+3}{6}} = 1$$

The length of each column of A is 1 and the dot product of any two different columns of A is 0.

So the columns of A form an orthonormal basis of $\mathbb{R}^3$.

So A is orthogonal.

So $A^{-1} = A^T$.

So $\vec{x} = A^{-1}\vec{b} = A^T\vec{b} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{3} \\ \sqrt{6} \\ \sqrt{2} \end{bmatrix}$

$$= \begin{bmatrix} 1+1+1 \\ 1-2+0 \\ 1+1-1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

$x_1 = 3$
$x_2 = -1$
$x_3 = 1$

7. (10 points) Let A be the matrix:

$$A = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix}$$

Compute $\lim_{t\to\infty} A^t$. (You must show your steps to receive credit.)

$$E_1 = \ker(A - I_2)$$
$$= \ker \begin{bmatrix} -\frac{3}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{3}{4} \end{bmatrix}$$
$$= \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

$$\vec{x}_{equ} = \frac{1}{1+1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\lim_{t\to\infty} A^t = \begin{bmatrix} | & | \\ \vec{x}_{equ} & \vec{x}_{equ} \\ | & | \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

8. (10 points) Let A be the matrix:

$$A = \begin{bmatrix} \cos(\frac{\pi}{7}) & -\sin(\frac{\pi}{7}) \\ \sin(\frac{\pi}{7}) & \cos(\frac{\pi}{7}) \end{bmatrix}$$

(a) (4 points) What is $|\det(A)|$? (You must explain your reasoning to receive credit.)

$$\det A = \cos^2\left(\frac{\pi}{7}\right) - \left(-\sin^2\left(\frac{\pi}{7}\right)\right)$$

$$= \cos^2\left(\frac{\pi}{7}\right) + \sin^2\left(\frac{\pi}{7}\right) = 1$$

$$|\det A| = |1| = 1$$

(b) (2 points) Is A invertible? (You must explain your reasoning to receive credit.)

Yes, since $\det(A) \neq 0$.

(c) (4 points) Is A diagonalizable over $\mathbb{R}$? (You must explain your reasoning to receive credit.)

No, because A is the matrix of rotation by $\frac{\pi}{7}$, which does not preserve the direction of any nonzero vector in $\mathbb{R}^2$ (i.e. $A\vec{v}$ is not parallel to $\vec{v}$ for any nonzero $\vec{v} \in \mathbb{R}^2$)

9. (10 points) A 3×3 matrix, A, has eigenvectors $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 7 \\ 2 \\ 1 \end{bmatrix}$ all of which have eigenvalue -1.

Find A. Simplify to a single 3×3 matrix. (You must show your steps to receive credit.)

$$A = SBS^{-1}$$
$$= S(-I_3)S^{-1}$$
$$= -SI_3S^{-1}$$
$$= -SS^{-1}$$
$$= -I_3$$
$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$
$$= -I_3$$

10. (10 points) Let $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ be basis of $\mathbb{R}^3$ with:

$$\begin{bmatrix}-1\\0\\4\end{bmatrix}_{\mathcal{B}} = \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\4\\2\end{bmatrix}_{\mathcal{B}} = \begin{bmatrix}0\\1\\0\end{bmatrix}, \text{ and } \begin{bmatrix}4\\3\\-1\end{bmatrix}_{\mathcal{B}} = \begin{bmatrix}-1\\0\\2\end{bmatrix}.$$

Find $\vec{v}_1, \vec{v}_2$,and $\vec{v}_3$. (You must show your steps to receive credit.)

$$\begin{bmatrix}-1\\0\\4\end{bmatrix} = \vec{v}_1 - \vec{v}_3 \qquad \begin{bmatrix}1\\4\\2\end{bmatrix} = \vec{v}_2 \qquad \begin{bmatrix}4\\3\\-1\end{bmatrix} = -\vec{v}_1 + 2\vec{v}_3$$

$$\begin{bmatrix}-1\\0\\4\end{bmatrix} + \begin{bmatrix}4\\3\\-1\end{bmatrix} = \vec{v}_1 - \vec{v}_3 - \vec{v}_1 + 2\vec{v}_3$$

$$\begin{bmatrix}3\\3\\3\end{bmatrix} = \vec{v}_3$$

$$\begin{bmatrix}-1\\0\\4\end{bmatrix} + \vec{v}_3 = \vec{v}_1$$

$$\begin{bmatrix}-1\\0\\4\end{bmatrix} + \begin{bmatrix}3\\3\\3\end{bmatrix} = \vec{v}_1$$

$$\begin{bmatrix}2\\3\\7\end{bmatrix} = \vec{v}_1$$