MATH 33A
SECOND MIDTERM EXAMINATION

May 20th, 2013

Please show your work. You will receive little or no credit for a correct answer to a problem which is not accompanied by sufficient explanations. If you have a question about any particular problem, please raise your hand and one of the proctors will come and talk to you. At the completion of the exam, please hand the exam booklet to your TA. If you have any questions about the grading of the exam, please see the instructor *within 15 calendar days of the examination (i.e. by June 4th).*

I certify that the work appearing on this exam is completely my own:
Signature: ____________________

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#1	#2	#3	#4	#5	Total
9	8	10	10	9	46

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Problem 1. Let V be the subspace of $\mathbb{R}^4$ consisting of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ such that

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 - x_2 + x_3 + x_4 &= 0 \end{aligned}$$

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(1) Find a basis for V.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 2 & 2 \\ 0 & -2 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$x_1 = -x^3 - x^4 = -s - t$

$x^2 = 0$

$x^3 = s$

$x^4 = t$

$$\boxed{\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}}$$

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(2) Find the dimension of V.

dimension is total number of vectors in a ~~subspace~~ basis

$\therefore 2$

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(3) Find an orthonormal basis for ~~B~~. V

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad u_1 \cdot v_2 = \frac{1}{\sqrt{2}}$$

$$v_{2\perp} = v_2 - (u_1 \cdot v_2) u_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$u_2 = \frac{v_{2\perp}}{\|v_{2\perp}\|} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\therefore \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$2012 - \frac{2013}{2} = \frac{4024}{2} - \frac{2013}{2}$ $\quad -4024 + \frac{6037}{2}$ $\quad \frac{8046}{2}$ $\quad 8048 - 6039 = 2009$

Problem 2. (10 pts) Let $\mathfrak{B}$ be the basis for $\mathbb{R}^2$ given by the vectors $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

(1) Let $w_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $w_2 = \begin{bmatrix} 2012 \\ 2013 \end{bmatrix}$ be two vectors in $\mathbb{R}^2$. Find the coordinates ${}_{\mathfrak{B}}[w_1]$ and ${}_{\mathfrak{B}}[w_2]$ of w_1 and w_2 in the basis $\mathfrak{B}$.

$S = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ $\quad S^{-1} = \frac{1}{4-6}\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}$

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$S^{-1}w_1 = B[w_1] = \begin{bmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \boxed{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$

$S^{-1}w_2 = B[w_2] = \begin{bmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 2012 \\ 2013 \end{bmatrix} = \boxed{\begin{bmatrix} -\frac{2009}{2} \\ \frac{2011}{2} \end{bmatrix}}$

(2) Let T be the transformation of $\mathbb{R}^2$ given by the matrix $A = [T] = \begin{bmatrix} 7 & -3 \\ 8 & -3 \end{bmatrix}$ in the standard basis. Find the matrix ${}_{\mathfrak{B}}[T]$ of T in the basis $\mathfrak{B}$.

$SB = AS$ $\quad 7-4 \quad 6-\frac{9}{2}$ $\quad -2+3$

$-3+\frac{3}{2}$ $\quad -6+6$

$B = S^{-1}AS$ $\quad 3-1$

$9-2$

$B = \begin{bmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 7 & -3 \\ 8 & -3 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$-6+6$

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$= \begin{bmatrix} -2 & \frac{3}{2} \\ 3 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 \\ 2 & \cancel{7} \end{bmatrix}}$ $\quad -2 \quad \frac{6}{2} \quad 9-2$

$\therefore {}_{B}[T] = \begin{bmatrix} 1 & 0 \\ 2 & \cancel{7} \end{bmatrix}$

Problem 3. (10 points)

Let I be the 5×5 identity matrix. Is there a 5×5 matrix A for which $A^T A = -I$ (here A^T denotes the transpose of A)? Either construct an example of such a matrix A, or explain why such a matrix can not exist. (The entreis of A can be arbitrary real numbers).

$A^T = A^{-1}$, $A^T A = I$

for all square matrix, $\det(A^T) = \det(A)$

$\det(A^T A) = \det(-I)$

$\det(A) \cdot \det(A) = \det(-I)$

$$-I = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$\det[-I] = -1$.

however, there is no matrix that could satisfy this condition because $\det(A^T A) = \det(A^T) \cdot \det(A) = \det(A) \cdot \det(A) = (\det(A))^2$

is always a positive number or a zero.

(it cannot be -1).

Therefore, no such matrix exists.

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Problem 4. (10 points) Find the least squares solution x^* of the system $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}.$$

$$x^* = (A^TA)^{-1}A^Tb$$

$$A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{4-1}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$(A^TA)^{-1} \cdot A^T = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$(A^TA)^{-1} \cdot A^Tb = \frac{1}{3}\begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\therefore\ x^* = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Problem 5. (True/False, 1 pt each) Mark your answers by filling in the appropriate box next to each question.

(a T F Let A be an orthogonal 4×4 matrix such that $Ae_1 = e_2$, $Ae_2 = e_3$ and $Ae_3 = e_1$. Then $Ae_4 = e_4$.

(b T F Let $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ be a set of linear independent vectors in a linear subspace V of $\mathbb{R}^n$. Then no two vectors $\mathbf{u}, \mathbf{w} \in V$ can span V.

(c T F If A is an orthogonal matrix, all the entries of A are less than or equal to 1.

(d T F The dimension of the vector space of all symmetric 4×4 matrices is 10.

(e T F Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be a linear transformation such that $\ker(T)$ has dimension 2. Then dimension of the image(T) is equal to 1. Image dimension: 3

(f T F If determinant of a matrix is equal to 1, the matrix is similar to the identity matrix.

(g T F Orthogonal projection is an example of an orthogonal transformation.

(h T F A linear transformation which takes an orthonormal basis into another orthonormal basis is orthogonal. preserves 90°, preserves length of 1

(i T F For any invertible matrix we have $(A^T)^{-1} = (A^{-1})^T$. ← formula

(j T F For any matrix A one can find Q and R such that $A = Q \cdot R$, where Q is an orthogonal matrix and R is an upper triangular matrix with non-negative entries on the diagonal.

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$