33A - Midterm 1

Name: Danny Nguyen

UID: 004628653

Question 1

(a) *[3 points]* Define the span of a set of vectors.

(b) *[4 points]* Explain whether or not the vector $\vec{x} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$ is in the span of the vectors $\vec{y} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$ and $\vec{z} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$.

(c) *[3 points]* Suppose that $\vec{v}_1$ is in the span of $\vec{v}_2, \vec{v}_3$. Does this imply that $\vec{v}_3$ is in the span of $\vec{v}_1, \vec{v}_2$?

....

a) The span of a set of vectors is the set of all possible linear combinations of those vectors.

$\Rightarrow \text{span}(\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n) = \{ w \mid w = a_1\bar{v}_1 + a_2\bar{v}_2 + \ldots + a_n\bar{v}_n \}$

b) $\begin{vmatrix} 1 & 1 & 1 \\ -1 & -2 & -2 \\ 4 & 0 & 4 \end{vmatrix} \to \begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 4 & 0 \end{vmatrix} \to \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{vmatrix} \to \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$

$\to \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

So this matrix has an rref $= I_3 \Rightarrow$ the three vectors must be linearly independent.

$\Rightarrow \bar{x} \notin \text{span}(\bar{y}, \bar{z})$

c) No. Let $\bar{v}_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \bar{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \bar{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\bar{v}_1 \in \text{span}(v_2, v_3)$ because

$\bar{v}_1 = 2\bar{v}_2 + 0\bar{v}_3$,

but $\bar{v}_3 \notin \text{span}(\bar{v}_1, \bar{v}_2)$. v_3 cannot be expressed as a linear combination of v_1 and v_2 since it has a x_2 component that is non-zero,

Question 2

(a) *[3 points]* Explain geometrically what it means to project a vector in 2D onto a line L.

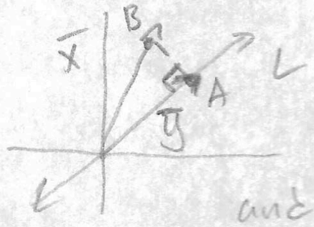
(b) *[4 points]* Find the projection of the vector $\vec{x} = \begin{pmatrix}1\\2\end{pmatrix}$ onto the line $L = span\left(\begin{pmatrix}1\\1\end{pmatrix}\right)$.

(c) *[3 points]* Use the projected vector you found in (b) to find the reflection of $\vec{x}$ in the line L. (An incorrect answer from (b) will not cost points here.)

....

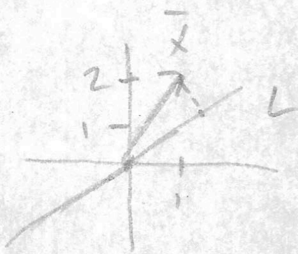
a) In 2D, projecting a vector onto a line means finding a second vector that is parallel to the line L on line L such that there is a segment perpendicular to L that has endpoints at the tails of the first and second vectors.

Ex:



$\bar{x}$ is original vector
$\bar{y}$ is vector on $L \Rightarrow \bar{y} \parallel L$
and $\overline{AB} \perp L$. Thus, $\mathrm{proj}_L \bar{x} = \bar{y}$.

b)



Let $\bar{u} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$

Then $|\bar{u}| = 1$

So. $\mathrm{Proj}_L \bar{x} = (\bar{u}\cdot\bar{x})\cdot\bar{u}$

$= \left[\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}\cdot\begin{pmatrix}1\\2\end{pmatrix}\right]\cdot\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$

$= \frac{3}{\sqrt{2}}\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix} = \boxed{\frac{3}{2}\begin{pmatrix}1\\1\end{pmatrix}}$

$A = \begin{bmatrix} u_1^2 & u_1u_2 \\ u_2u_2 & u_2^2 \end{bmatrix}$

$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}$

$= \begin{bmatrix}3/2\\3/2\end{bmatrix}$

c) Reflections can be expressed as $(2A_{proj} - I_n)\bar{x}$

$$= \begin{bmatrix} 2u_1^2 - 1 & 2u_1u_2 - 0 \\ 2u_1u_2 & 2u_2^2 - 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}$$

Question 3

(a) *[2 points]* Define the kernel of a matrix.

(b) *[3 points]* Show that for any $m \times n$ matrix A, saying that the $\ker(A) = \{\vec{0}\}$ is the same as saying that the columns of A are independent.

(c) *[3 points]* Show that if $\vec{x}, \vec{y} \in \ker(A)$, and $\vec{z} \in \operatorname{span}(\vec{x}, \vec{y})$, then $\vec{z} \in \ker(A)$.

(d) *[2 points]* Do elementary row operations preserve the kernel of a matrix?

....

a) The kernel of a matrix A is the set of all vectors that when multiplied by A yields 0.

$$\ker(A) = \{ \bar{x} \mid A\bar{x} = \bar{0} \} \quad 2$$

b) Let $\ker(A) = \{0\}$. So $\forall x \in \mathbb{R}^n$, $A\bar{x} \neq \bar{0}$ if $\bar{x} \neq 0$.

Let $A = \begin{bmatrix} | & | & | & & | \\ v_1 & v_2 & v_3 & \cdots & v_n \\ | & | & | & & | \end{bmatrix}$, $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ 3

then $A\bar{x} = \bar{v}_1 x_1 + \bar{v}_2 x_2 + \bar{v}_3 x_3 + \ldots \bar{v}_n x_n \neq 0$ if $\bar{x} \neq 0$.

So the only solution to $A\bar{x} = 0$ is the trivial solution, so $\bar{v}_1, \bar{v}_2, \bar{v}_3, \ldots, \bar{v}_n$, or the columns of A, must be linearly independent.

c) This is the same as showing that $\ker(A)$ is closed under addition and scalar multiplication.

If $(x, y) \in \ker(A)$, $A\bar{x} = 0 = A\bar{y}$.

If $\bar{z} \in \operatorname{span}(\bar{x}, \bar{y})$, $\bar{z} = \alpha\bar{x} + \beta\bar{y}$ where α, β are simple scalars 3

Then $A\bar{z} = A(\alpha\bar{x} + \beta\bar{y}) = A(\alpha\bar{x}) + A(\beta\bar{y})$
$= \alpha(A\bar{x}) + \beta(A\bar{y})$ } By properties of linear transformations
$= \alpha\bar{0} + \beta\bar{0} = \bar{0}$

So $A\bar{z} = 0 \Rightarrow \bar{z} \in \ker(A)$.

d) Yes. The kernel of a matrix is the set of vectors that is perpendicular to the row-span of the matrix. Since elementary row operations do not alter the row-span (because span(basis for row-span RREF(A) $\subseteq$ span(basis for row-span of A), the kernel of this matrix is therefore preserved.