Question 1

A subspace V of $\mathbb{R}^n$ is called a *hyperplane* if the vectors $\vec{x} \in V$ are defined by an equation: $a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$, where at least one of the coefficients a_i is nonzero.

(a) *[3 points]* How many of the variables x_i are free? What is the dimension of a hyperplane in $\mathbb{R}^n$?

(b) *[4 points]* Explain what a hyperplane in $\mathbb{R}^2$ looks like, and give a basis for the hyperplane in $\mathbb{R}^2$ given by the equation $x_1 + 2x_2 = 0$.

(c) *[3 points]* Find an equation like the one given above for the plane spanned by the vectors $\begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\0\\0\end{pmatrix}$ in $\mathbb{R}^3$.

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Answer

(a) If all the $a_i = 0$, then the equation would just say $0 = 0$ and all the variables are free. Since this is not the case, in order for the equation $a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$ to hold, we pick one variable x_i with a nonzero a_i. Then all the other variables are free, and they determine the value of x_i. Since there are $n - 1$ free variables, we can write our vectors in V as linear combinations of exactly $n - 1$ vectors, so V has $n - 1$ dimensions.

(b) The equation $x_1 + 2x_2 = 0$ gives us $x_2 = s$ and $x_1 = -2s$, so the vectors look like $s\begin{pmatrix}-2\\1\end{pmatrix}$. We have a 1-dimensional subspace (a line) with basis $\begin{pmatrix}-2\\1\end{pmatrix}$.

(c) All vectors in this space can be written as $\vec{x} = s\begin{pmatrix}1\\1\\0\end{pmatrix} + t\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}s+t\\t\\0\end{pmatrix}$. This means that x_1 and x_2 are free, and $x_3 = 0$: that's our equation. (Those two vectors clearly span the horizontal plane in 3D where the third entry of each vector is 0.)

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Question 2

(a) *[3 points]* Define the kernel and image of a matrix A.

(b) *[4 points]* Verify that both the kernel and image of A are closed under addition and scalar multiplication.

(c) *[3 points]* Find a 3×3 matrix A such that both the kernel and the image of A contain $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

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Answer

(a) The kernel of a matrix A is the set of all vectors $\vec{x}$ in the domain of A such that $A\vec{x} = \vec{0}$. The image of A is the set of all vectors $\vec{y}$ in the target spaces of A such that there exists an $\vec{x}$ in the domain for which $A\vec{x} = \vec{y}$.

(b) For the kernel:

If $\vec{x}_1, \vec{x}_2 \in \ker(A)$, then $A(\vec{x}_1 + \vec{x}_2) = A\vec{x}_1 + A\vec{x}_2 = \vec{0}$. Therefore $\vec{x}_1 + \vec{x}_2$ is in the kernel: closed under addition. If $\vec{x} \in \ker(A)$, then $A(k\vec{x}) = kA\vec{x} = k\vec{0} = \vec{0}$. Therefore, $k\vec{x}$ is in the kernel, for any $k \in \mathbb{R}$: closed under scalar multiplication.

For the image:

If $\vec{y}_1, \vec{y}_2 \in \operatorname{im}(A)$, then there exist $\vec{x}_1, \vec{x}_2$ (not necessarily the same vectors as earlier) such that $A\vec{x}_1 = \vec{y}_1$ and $A\vec{x}_2 = \vec{y}_2$. Then $A(\vec{x}_1 + \vec{x}_2) = A\vec{x}_1 + A\vec{x}_2 = \vec{y}_1 + \vec{y}_2$. Therefore $\vec{y}_1 + \vec{y}_2 \in \operatorname{im}(A)$: closed under addition. If $\vec{y} \in \operatorname{im}(A)$, then there exists some $\vec{x}$ in the domain of A for which $A\vec{x} = \vec{y}$. Then $Ak\vec{x} = kA\vec{x} = k\vec{y}$, so $k\vec{y} \in \operatorname{im}(A)$, for any $k \in \mathbb{R}$: closed under scalar multiplication.

(c) A for $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ to be in the kernel, we must have that each row of A satisfies $x_1 + 2x_2 + 3x_3 = 0$. This means that for each row we have two free variables. For $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ to be in the image, we can just add it in as one of the columns. Since we have two free variables per row we can add it in as a column twice, and then find the third entries for each row:

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 3 & 3 & -3 \end{pmatrix}.$$

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Question 3

(a) *[3 points]* Define what it means for a matrix to be invertible.

(b) *[4 points]* Find the projection of the vector $\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ onto the line L spanned by $\vec{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Explain why this transformation is not invertible.

(c) *[3 points]* For which values of x is the following matrix invertible? First answer this question using the determinant of A, and then give a geometrical interpretation.

$$A = \begin{pmatrix} 1 & a \\ -a & 1 \end{pmatrix}$$

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Answer

(a) A square matrix A is invertible if there exists a matrix A^{-1} such that $AA^{-1} = I_n$. (There are many other possible definitions in terms of kernels, ranks, determinants, etc...)

(b) We first find the unit vector in the direction on L. We have $\vec{u} = \frac{\vec{w}}{||\vec{w}||} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Now $proj_L \vec{x} = (\vec{x} \cdot \vec{u})\vec{u} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. (This answer is also easy to deduce from a sketch.) Alternatively, you can use the matrix for the 2-D projection in terms of u_1 and u_2 that we defined in the lectures.

(c) We have that $\det(A) = 1 + a^2 > 0$ for any $a \in \mathbb{R}$, so A is always invertible. We know that a matrix of the form

$$A = \begin{pmatrix} b & a \\ -a & b \end{pmatrix}$$

represents a rotation followed by a scaling. Both rotations and scalings are clearly invertible: a scaling by a factor k can be inverted by a scaling by $1/k$, and a rotation over an angle θ can be inverted by a rotation over an angle $-\theta$. Therefore, it is clear that A should indeed be an invertible matrix.

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