MATH 33A EXAM 2 — Name:

November 13, 2015 — SID:

Name of TA:

Discussion section:

I have read and understood the Student Honor Code, and this exam reflects my unwavering commitment to the principles of academic integrity and honesty expressed therein.

Signature:__________ ________________

Each part of each problem is worth the number of points stated in parentheses. You must show all work to get any partial credit, which will be awarded for certain progress in a problem only if no substantially false statements have been written.

Please write your answers to problems A-E in the box below.

Problem	Answer	Points
1A	number	1
1B	Subspace	1
1C	True	0
1D	5	2
1E	$\pi/4$	2
1F	A^{-1}	0

6

This side for instructor's use only:

Problem	Points
2a	6
2b	0
3a	6
3b	2
3c	0
4	0

6 (2a–2b), 8 (3a–3c), 0 (4)

14

6 + 14 = 20

Problem 1. For each question below, you are given a set of possible answers. Select the single answer which best answers the question, and write it on the blank provided. You must also write your answer on the front page of the exam in the box provided.

There is no partial credit on this problem.

number (A) (1 point) The rank of an $m \times n$ matrix is what type of thing?
{ number, vector, subspace, matrix, linear transformation }

Subspace (B) (1 point) The span of the columns of an $m \times n$ matrix is what type of thing?
{ number, vector, subspace, matrix, linear transformation }

True (C) (2 points) Suppose A is an $n \times n$ matrix such that $\langle \vec{x}, \vec{y} \rangle = 0$ implies $\langle A\vec{x}, A\vec{y} \rangle = 0$. Then A is an orthogonal matrix.
{ True, False }

5 (D) (2 points) Let A be a 6×7 matrix whose image is two dimensional. What is the dimension of $\mathrm{im}(A^T)^{\perp}$?
$\{0, 1, 2, 3, 4, 5, 6, 7\}$

$\pi/4$ (E) (2 points) What is the angle between the vectors $\begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix}$ and $\begin{pmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{pmatrix}$?

$$\left\{ \pi, \frac{5\pi}{6}, \frac{3\pi}{4}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}, 0 \right\}.$$

A^{-1} (F) (2 points) Suppose A and B are $n \times n$ symmetric invertible matrices. Which of the following is not necessarily symmetric?
$\{A^T, A^2, A^{-1}, A + B, A - B^T, AB\}$

Problem 2. (10 points) You must show all work to get partial credit. You must simplify your work to receive full credit.

(a) (8 points) Compute the QR factorization of the matrix $M = \begin{pmatrix} 1 & 0 & 1 \\ 7 & 7 & 8 \\ 1 & 2 & 1 \\ 7 & 7 & 6 \end{pmatrix}$.

$\vec{w}_1 = \begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix}$, $\vec{u}_1 = \frac{1}{10}\begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix}$

$\vec{w}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$\vec{w}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$, $\vec{u}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

$Q = \begin{bmatrix} 1/10 & -1/\sqrt{2} & 0 \\ 7/10 & 0 & 1/\sqrt{2} \\ 1/10 & 1/\sqrt{2} & 0 \\ 7/10 & 0 & -1/\sqrt{2} \end{bmatrix}$, $R = \begin{bmatrix} 10 & 22/5 & 18/5 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$

(b) (2 points) Compute the matrix of the projection onto $V = \mathrm{im}(M)$.

Problem 3. (10 points) You must show all work to get partial credit. You must simplify your work to receive full credit.

Consider $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$.

(a) (6 points) Find a vector $\vec{x}^*$ which minimizes $\|A\vec{x}^* - \vec{b}\|$.

$A^T A \vec{x}^* = A^T \vec{b}$

$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}^* = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$

$\left[\begin{array}{cc|c} 2 & 1 & 6 \\ 1 & 2 & 6 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 6 \\ 2 & 1 & 6 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 6 \\ 0 & -3 & -6 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 6 \\ 0 & 1 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 2 \end{array}\right]$

$x_1 = x_2 = 2$ ✓

6

$\boxed{\vec{x}^* = \begin{bmatrix} 2 \\ 2 \end{bmatrix}}$

(b) (2 points) Verify that $A\vec{x}^* - \vec{b}$ is orthogonal to $\text{im}(A)$.

$\text{im}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ ✓

$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

2

$\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 0 \qquad \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0$ ✓

$A\vec{x}^* - \vec{b}$ is orth to im (A)

(c) (2 points) Compute the matrix of the projection onto $\text{im}(A)$.

0

$\sim \times n \wedge \times p$

Problem 4. (10 points) Let A be an $m \times n$ matrix and $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \in \mathbb{R}^n$. Suppose that $A\vec{v}_1, A\vec{v}_2, A\vec{v}_3, A\vec{v}_4$ are linearly independent. Show that $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are linearly independent.

Let

$\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are $n \times 1$

let $S = \begin{bmatrix} | & | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \\ | & | & | & | \end{bmatrix}$

let

$T(\vec{x}) = A\vec{x}$

since $A\vec{v}_3$ is lin ind of $A\vec{v}_1$ and $A\vec{v}_2$

only $c_1 = c_2 = 0$ solves

$T(\vec{v}_3) = A\vec{v}_3 = c_1(A\vec{v}_1) + c_2(A\vec{v}_2)$

$= \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} T(\vec{v}_1) \\ T(\vec{v}_2) \end{bmatrix}$

if $A\vec{v}_1 \cdot A\vec{v}_2 = 0$

show $\vec{v}_1 \cdot \vec{v}_2 = 0$

let $\vec{v}_1 = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$

$A\vec{v}_1 = \begin{bmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & & \vdots \\ A_{M1} & \cdots & A_{MN} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix} = \begin{bmatrix} (A_{11}v_1 + A_{12}v_2 + \cdots A_{1N}v_N) \\ (A_{21}v_1 + \cdots + A_{2N}v_N) \\ \cdots \\ A_{M1}v_1 + \cdots + A_{MN}v_N \end{bmatrix}$

then

$A\vec{v}_1 \cdot A\vec{v}_2 = \begin{bmatrix} (A_{11}v_1 + \cdots + A_{1N}v_N) + (A_{11}u_1 + \cdots + A_{1N}u_N) \\ - \rightarrow - \\ (A_{M1}v_1 + \cdots + A_{MN}) + (A_{M1}u_1 + \cdots + A_{MN}u_N) \end{bmatrix}$

$\begin{bmatrix} v_1 \\ \\ \end{bmatrix}$