**Problem 1.**

__________(A) (1 point) The rank of an $m \times n$ matrix is what type of thing?
{ number, vector, subspace, matrix, linear transformation }
**Solution:** The rank is the number of pivots in the reduced row echelon form of the matrix. Thus is is a number.

__________(B) (1 point) The span of the columns of an $m \times n$ matrix is what type of thing?
{ number, vector, subspace, matrix, linear transformation }
**Solution:** The span is set of all linear combinations of the columns, which is a subspace.

__________(C) (2 points) Suppose $A$ is an $n \times n$ matrix such that $\langle \vec{x}, \vec{y} \rangle = 0$ implies $\langle A\vec{x}, A\vec{y} \rangle = 0$. Then $A$ is an orthogonal matrix.
{ True, False }
**Solution:** False. Consider the zero $n \times n$ matrix, which has zero as every entry.

__________(D) (2 points) Let $A$ be a $6 \times 7$ matrix whose image is two dimensional. What is the dimension of $\operatorname{im}(A^T)^\perp$?
$\{0, 1, 2, 3, 4, 5, 6, 7\}$
**Solution:** 5. One computes that $\operatorname{im}(A^T)^\perp = \ker((A^T)^T) = \ker(A)$. By the rank nullity theorem, $\dim(\ker(A)) + \dim(\operatorname{im}(A)) = 7$, the number of columns. Hence $\dim(\ker(A)) = 7 - 2 = 5$.
**Remark:** The other version of this test had a $7 \times 5$ matrix, for which the answer is 3.

__________(E) (2 points) What is the angle between the vectors $\begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix}$ and $\begin{pmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{pmatrix}$ ?

$\left\{\pi, \frac{5\pi}{6}, \frac{3\pi}{4}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}, 0\right\}$.
**Solution:** $\pi/4$. We compute that $\cos(\theta)$ is the dot product of the two vectors divided by their lengths. The dot product is $\sqrt{2}$, and their lengths are 1 and 2 respectively. Thus $\theta = \cos^{-1}(\sqrt{2}/2)$.
**Remark:** The other version of the test had the same answer, but the vectors were a bit different.

__________(F) (2 points) Suppose $A$ and $B$ are $n \times n$ symmetric invertible matrices. Which of the following is not necessarily symmetric?
$\{A^T, A^2, A^{-1}, A + B, A - B^T, AB\}$
**Solution:** If $A$ and $B$ are symmetric, then $A = A^T$ and $B = B^T$, so we

can write all the answers without transposes:
$\{A, A^2, A^{-1}, A+B, A-B, AB\}$
Now the map $A \mapsto A^T$ is a linear map, so we can reduce the choices to $\{A^2, A^{-1}, AB\}$
We know that $(A^{-1})^T = (A^T)^{-1} = A^{-1}$, so we can eliminate that option. Finally, $(AB)^T = B^T A^T$ and $(A^2)^T = A^T A^T = AA = A^2$. The answer is $AB$.
**Remark:** The other version of the test had different choices of answers: $\left\{B^2, B^{-1}, A^T B^T, A+B, A-B^T, B^T\right\}$
The correct answer here is the product $A^T B^T$.

**Problem 2.** (10 points)

(a) (8 points) Compute the $QR$ factorization of the matrix $M = \begin{pmatrix} 1 & 0 & 1 \\ 7 & 7 & 8 \\ 1 & 2 & 1 \\ 7 & 7 & 6 \end{pmatrix}$.

**Solution:** We compute the $QR$ factorization by first computing the Gram-Schmidt orthonormalization of the columns. The first column $\vec{v}_1 = \vec{w}_1$ has length 10, so

$$\vec{u}_1 = \frac{1}{10}\begin{pmatrix} 1 \\ 7 \\ 1 \\ 7 \end{pmatrix}.$$

We set $\vec{v}_2$ to be the second column, and

$$\vec{w}_2 = \vec{v}_2 - \langle \vec{v}_2, \vec{u}_1 \rangle \vec{u}_1 = \begin{pmatrix} 0 \\ 7 \\ 2 \\ 7 \end{pmatrix} - \frac{100}{10}\frac{1}{10}\begin{pmatrix} 1 \\ 7 \\ 1 \\ 7 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

This vector has length $\sqrt{2}$, so $\vec{u}_2 = \frac{1}{\sqrt{2}}\vec{w}_2$. Finally, we set $\vec{v}_3$ to be the third column and

$$\vec{w}_3 = \vec{v}_3 - \langle \vec{v}_3, \vec{u}_1 \rangle \vec{u}_1 - \langle \vec{v}_3, \vec{u}_2 \rangle \vec{u}_2 = \begin{pmatrix} 1 \\ 8 \\ 1 \\ 6 \end{pmatrix} - \frac{100}{10}\frac{1}{10}\begin{pmatrix} 1 \\ 7 \\ 1 \\ 7 \end{pmatrix} - 0\vec{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}.$$

Again, $\vec{w}_3$ has length $\sqrt{2}$, so $\vec{u}_3 = \frac{1}{\sqrt{2}}\vec{w}_3$. This means that

$$Q = \begin{pmatrix} \frac{1}{10} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{7}{10} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{10} & \frac{1}{\sqrt{2}} & 0 \\ \frac{7}{10} & 0 & \frac{-1}{\sqrt{2}} \end{pmatrix} \qquad \text{and} \qquad M = \begin{pmatrix} 10 & 10 & 10 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

(b) (2 points) Compute the matrix of the projection onto $V = \mathrm{im}(M)$.
**Solution:** The matrix of the projection is

$$QQ^T = \begin{pmatrix} \frac{1}{10} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{7}{10} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{10} & \frac{1}{\sqrt{2}} & 0 \\ \frac{7}{10} & 0 & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{10} & \frac{7}{10} & \frac{1}{10} & \frac{7}{10} \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{pmatrix} = \frac{1}{100} \begin{pmatrix} 51 & 7 & -49 & 7 \\ 7 & 99 & 7 & -1 \\ -49 & 7 & 51 & 7 \\ 7 & -1 & 7 & 99 \end{pmatrix}.$$

**Problem 3.** (10 points) Consider $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$.

(a) (6 points) Find a vector $\vec{x}^*$ which minimizes $\|A\vec{x}^* - \vec{b}\|$.
**Solution:** We compute

$$A^T A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

which is invertible. Hence

$$\vec{x}^* = (A^T A)^{-1} A^T \vec{b} = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

(b) (2 points) Verify that $A\vec{x}^* - \vec{b}$ is orthogonal to $\mathrm{im}(A)$.
**Solution:** We compute

$$A\vec{x}^* - \vec{b} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

which is easily seen to have zero inner product with the columns of $A$.

(c) (2 points) Compute the matrix of the projection onto $\mathrm{im}(A)$.
**Solution:** We calculate

$$A(A^T A)^{-1} A^T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

**Problem 4.** (10 points) Let $A$ be an $m \times n$ matrix and $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \in \mathbb{R}^n$. Suppose that $A\vec{v}_1, A\vec{v}_2, A\vec{v}_3, A\vec{v}_4$ are linearly independent. Show that $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are linearly independent.
**Solution:** Suppose we have a linear relation $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 = \vec{0}$. We must show it is is the trivial relation, i.e., $c_1 = c_2 = c_3 = c_4 = 0$. Left multiply by $A$ to get the linear relation $c_1A\vec{v}_1 + c_2A\vec{v}_2 + c_3A\vec{v}_3 + c_4A\vec{v}_4 = \vec{0}$. But $A\vec{v}_1, A\vec{v}_2, A\vec{v}_3, A\vec{v}_4$ are linearly independent, so the only linear relation is the trivial linear relation, i.e., $c_1 = c_2 = c_3 = c_4 = 0$.