Please write your answers to problems A-E in the box below.

Problem	Answer	Points
1A	TRUE	0
1B	FALSE	2
1C	FALSE	2
1D	FALSE	2
1E	FALSE	0

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Problem 1. Below is a list of statements. Decide which are true and which are false. On the left of each, write "TRUE" or "FALSE" in capital letters. You must also write your answer ("TRUE" or "FALSE" in capital letters) on the front page of the exam.

There is no partial credit on this problem.

TRUE (A) (2 points) Suppose A is $m \times n$. The vector $A\vec{x}$ for $\vec{x} \in \mathbb{R}^n$ is a linear combination of the rows of A.

$$\begin{bmatrix} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n \\ \vdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n \end{bmatrix} \leftarrow \text{lin. comb of r}$$

FALSE (B) (2 points) Every square matrix is invertible.

FALSE (C) (2 points) Suppose A is $m \times n$, and $A\vec{x} = \vec{b}$ has a unique solution for some vector $\vec{b} \in \mathbb{R}^m$. Then $n > m$.

$n \leq m$

van. sexns.

$n \leq m$

FALSE (D) (2 points) Suppose A is $m \times n$, with $n > m$. Then $\ker(A) = \{0\}$.

bijective

FALSE (E) (2 points) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one, then $\ker(T) = \{\vec{0}\}$.

Problem 2. (*10 points*) You must show all work to get partial credit.

(a) (5 points) Use Gaussian elimination to compute the inverse of $A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$.

$$(A \vdots I_n) = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right] \xrightarrow{-3R_1+R_2 \to R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -3 & 1 \end{array}\right] \xrightarrow[-R_2 \to R_2]{2R_2+R_1 \to R_1} \begin{array}{c} I_n \;\vdots\; A^{-1} \\ \left[\begin{array}{cc|cc} 1 & 0 & -5 & 2 \\ 0 & 1 & 3 & -1 \end{array}\right] \end{array}$$

$$\boxed{A^{-1} = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}}$$

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(b) (5 points) Find a 2×3 matrix A and a 3×2 matrix B such that $AB = I_2$, but $BA \neq I_3$.
Hint: You can do this only using 0's and 1's for the entries of A and B.

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$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix} \qquad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix}$$

$$AB = \begin{bmatrix} \boxed{A_{11}B_{11}} + A_{12}B_{21} + A_{13}B_{31} & A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32} \\ A_{21}B_{11} + A_{22}B_{21} + A_{23}B_{31} & A_{21}B_{12} + A_{22}B_{22} + \boxed{A_{23}B_{32}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} \boxed{B_{11}A_{11}} + B_{12}A_{21} & B_{11}A_{12} + B_{12}A_{22} & B_{11}A_{13} + B_{12}A_{23} \\ B_{21}A_{11} + B_{22}A_{21} & B_{21}A_{12} + B_{22}A_{22} & B_{21}A_{13} + B_{22}A_{23} \\ B_{31}A_{11} + B_{32}A_{21} & B_{31}A_{12} + B_{32}A_{22} & B_{31}A_{13} + \boxed{B_{32}A_{23}} \end{bmatrix}$$

if $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$

then $\underset{I_2}{AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$ and $BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq I_3$

(looked at boxed pairs w/ underlines)

Problem 3. (*10 points*) You must show all work to get partial credit.

Consider the 4×5 matrix $A = \begin{pmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

(a) (5 points) Find a set of vectors in $\mathbb{R}^5$ which spans $\ker(A)$.

$x_2 = s$ & $x_5 = t$

$A\vec{x} = 0 \quad \Rightarrow x_1 = -2s-3t, \; x_3 = -2t, \; x_4 = -t$

$$\vec{x} = s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ -2 \\ -1 \\ 1 \end{bmatrix}$$

$$\ker(A) = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -2 \\ -1 \\ 1 \end{bmatrix} \right\}$$

(b) (3 points) Find a set of vectors in $\mathbb{R}^4$ which spans $\text{im}(A)$.

$\text{im}(A) =$ columns w/o free variables

$$= \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

(c) (2 points) If you answered both part (a) and part (b) correctly, you get the final 2 points for this question if the number of vectors you found in part (a) plus the number of vectors you found in part (b) equals 5.

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Problem 4. (*10 points*) Suppose we have j vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_j \in \mathbb{R}^n$ and k vectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_j\}$. Prove that any linear combination of $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ is also a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_j$.

Note: You are being asked to prove $V = \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_j\}$ is closed under taking linear combinations. You may not assume V is a subspace in this question, since that would beg the question!

Because $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k \in \text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_j\}$, $w_i = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_j\vec{v}_j$ where $1 \le i \le k$.
(The i^{th} w vector is a linear combination of the v vectors). $\left[w_i = \sum_{n=1}^{j} c_n\vec{v}_n \right]$.

A linear combination of all $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ can be written as
$\sum_{i=1}^{k} d_i w_i = \sum_{i=1}^{k} \sum_{n=1}^{j} c_n\vec{v}_n$ since the sum of scalars is also a scalar say c_m we have a linear comb. of $\vec{v}_1, \vec{v}_2, \vec{v}_j$

$\vec{w}_1 = c_{11}\vec{v}_1 + \cdots + c_{1j}\vec{v}_j$

$\vec{w}_2 = c_{21}\vec{v}_1 + \cdots + c_{2j}\vec{v}_j$

$\vdots$

$\vec{w}_K = c_{k1}\vec{v}_1 + \cdots + c_{kj}\vec{v}_j$

$d_1\vec{w}_1 + d_2\vec{w}_2 + \cdots + d_k\vec{w}_K = (c_{11}\vec{v}_1 + c_{21}\vec{v}_1 + \cdots + c_{k1}\vec{v}_1) + \cdots + (c_{1j}\vec{v}_j + c_{2j}\vec{v}_j + \cdots + c_{kj})$

$= \underbrace{(c_{11} + c_{21} + c_{k1})}_{\text{scalar}}\vec{v}_1 + \cdots + \underbrace{(c_{1j} + c_{2j} + \cdots + c_{kj})}_{\text{Scalar}}\vec{v}_j$

$= c_1\vec{v}_1 + \cdots + c_j\vec{v}_j$ ← linear comb

Also if any $\vec{w}_i$ vector is excluded, we can write this as $c_i = 0$ and $c_1 + c_2 + \cdots + c_i + \cdots + c_k$ is still a scalar & the end result is still a linear comb