MATH 33A EXAM 1

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I have read and understood the Student Honor Code, and this exam reflects my unwavering commitment to the principles of academic integrity and honesty expressed therein.

Signature:

Each part of each problem is worth the number of points stated in parentheses. You must show all work to get any partial credit, which will be awarded for certain progress in a problem only if no substantially false statements have been written.

Please write your answers to problems A-E in the box below.

Problem	Answer	Points
1A	FALSE	2
1B	FALSE	2
1C	FALSE	2
1D	FALSE	2
1E	FALSE	0

8

This side for instructor's use only:

Problem	Points
2a	5
2b	5
3a	5
3b	3
3c	2
4	10

30

38

Problem 1. Below is a list of statements. Decide which are true and which are false. On the left of each, write "TRUE" or "FALSE" in capital letters. You must also write your answer ("TRUE" or "FALSE" in capital letters) on the front page of the exam.

There is no partial credit on this problem.

FALSE (A) (2 points) Suppose A is $m \times n$. The vector $A\vec{x}$ for $\vec{x} \in \mathbb{R}^n$ is a linear combination of the rows of A.

columns

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_2$$

FALSE (B) (2 points) Every square matrix is invertible. LOL You think?

FALSE (C) (2 points) Suppose A is $m \times n$, and $A\vec{x} = \vec{b}$ has a unique solution for some vector $\vec{b} \in \mathbb{R}^m$. Then $n > m$.

m n

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

free var.

FALSE (D) (2 points) Suppose A is $m \times n$, with $n > m$. Then $\ker(A) = \{0\}$.

cols > rows

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

FALSE (E) (2 points) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one, then $\ker(T) = \{\vec{0}\}$.

one-to-one
and onto.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 7 \end{bmatrix}$$

test: $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} -5+6=1 & 2-2=0 \\ -15+15=0 & 6-5=1 \end{bmatrix}$

Problem 2. (*10 points*) You must show all work to get partial credit.

(a) (5 points) Use Gaussian elimination to compute the inverse of $A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$.

$$\begin{array}{c} ① \\ ② \end{array}\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} ① \\ ②-3\times① \end{array}\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3-3 & 5-6 & 0-3 & 1 \end{array}\right]$$

$$\begin{array}{c} ① \\ ② \end{array}\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -3 & 1 \end{array}\right]$$

$$\begin{array}{c} \\ -② \end{array}\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3 & -1 \end{array}\right]$$

$$\begin{array}{c} ①-2② \\ \end{array}\left[\begin{array}{cc|cc} 1 & 0 & 1-6 & 0+2 \\ 0 & 1 & 3 & -1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -5 & 2 \\ 0 & 1 & 3 & -1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$$

(b) (5 points) Find a 2×3 matrix A and a 3×2 matrix B such that $AB = I_2$, but $BA \neq I_3$.
Hint: You can do this only using 0's and 1's for the entries of A and B.

$$A\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \quad B\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \\ b_5 & b_6 \end{bmatrix}$$

~~AB~~

$$AB = \overset{A}{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}\overset{B}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$BA = \overset{B}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}\overset{A}{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \neq I_3$$

$$\boxed{A: \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^3 \quad B: \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}$$

Problem 3. (*10 points*) You must show all work to get partial credit.

Consider the 4×5 matrix $A = \begin{pmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

(a) (5 points) Find a set of vectors in $\mathbb{R}^5$ which spans $\ker(A)$.

$$A = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

let $x_5 = t$, $x_2 = s$

$x_4 = 0 - t$

$x_3 = -2t$

$x_2 = s$

$x_1 = -3t - 2s$

then $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3t-2s \\ s \\ -2t \\ -t \\ t \end{bmatrix}$

$= t \begin{bmatrix} -3 \\ 0 \\ -2 \\ -1 \\ 1 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\ker(A) = \text{span}\left\{ \begin{bmatrix} -3 \\ 0 \\ -2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

(b) (3 points) Find a set of vectors in $\mathbb{R}^4$ which spans $\text{im}(A)$.

$\text{im}(A) = \{\text{lin. comb. of the col's of } A\}$

$= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

note: $\begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$ is lin. comb. of $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is lin. comb of $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

(c) (2 points) If you answered both part (a) and part (b) correctly, you get the final 2 points for this question if the number of vectors you found in part (a) plus the number of vectors you found in part (b) equals 5.

4

$2 + 3 = 5$ yes!

10

Problem 4. (*10 points*) Suppose we have j vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_j \in \mathbb{R}^n$ and k vectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_j\}$. Prove that any linear combination of $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ is also a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_j$.

Note: You are being asked to prove $V = \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_j\}$ is closed under taking linear combinations. You may not assume V is a subspace in this question, since that would beg the question!

$\because$ $\vec{w}_1, \vec{w}_2 \ldots \vec{w}_k \in \text{span}\{\vec{v}_1 \ldots \vec{v}_j\}$,

$\therefore$ $\vec{w}_1, \vec{w}_2, \ldots \vec{w}_k$ are linear combinations of $\vec{v}_1 \ldots \vec{v}_j$.

then, $\vec{w}_1 = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots c_j\vec{v}_j$

$\vec{w}_2 = a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots a_j\vec{v}_j$

$\vdots$

$\vec{w}_k = b_1\vec{v}_1 + b_2\vec{v}_2 + \ldots b_j\vec{v}_j$ (c_i, a_i, b_i are some constants)

let $\vec{x}$ be lin. comb. of $\vec{w}_1, \vec{w}_2 \ldots \vec{w}_k$,

then $\vec{x} = h_1\vec{w}_1 + h_2\vec{w}_2 + \ldots h_k\vec{w}_k$

$= h_1(c_1\vec{v}_1 + \ldots c_j\vec{v}_j) + h_2(a_1\vec{v}_1 + \ldots a_j\vec{v}_j) + h_k(b_1\vec{v}_1 + \ldots b_j\vec{v}_j)$

$= \underbrace{(h_1c_1 + h_2a_1 + \ldots h_kb_1)}_{\text{some constant } p_1}\vec{v}_1 + \underbrace{(h_1c_2 + h_2a_2 + h_kb_2)}_{\text{some constant } p_2}\vec{v}_2 \ldots + \underbrace{(h_1c_j + h_2a_j + h_ka_j)}_{\text{some constant } p_j}\vec{v}_j)$

$= p_1\vec{v}_1 + p_2\vec{v}_2 + p_2\vec{v}_3 + \ldots p_j\vec{v}_j$ proven

therefore,

any linear comb. of $\vec{w}_1, \vec{w}_2 \ldots \vec{w}_k$ is also a linear comb. of $\vec{v}_1, \vec{v}_2 \ldots \vec{v}_j$