1. (25 points) Let $T(\mathbf{x}) = A\mathbf{x}$ a linear transformation w/ $A$ def'd by:

$$A = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 4 \end{bmatrix} \qquad \text{If } \mathbf{b} = \begin{bmatrix} -8 \\ 6 \\ 0 \\ 12 \end{bmatrix}$$

Then find all solutions of the system $A\mathbf{x} = \mathbf{b}$. What is the Rank of the Linear Transformation? (You may use the back of this sheet if you need additional space.)

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 3 & 4 & -6 & 8 & 0 \\ 0 & -1 & 3 & 4 & -12 \end{array}\right) \xrightarrow[-3R_1+R_3 \to R_3]{} \left(\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 4 & -12 & -4 & 24 \\ 0 & -1 & 3 & 4 & -12 \end{array}\right) \xrightarrow{\substack{-4R_2+R_3 \to R_3 \\ R_2+R_4 \to R_4}} \left(\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & -6 \end{array}\right)$$

$$\xrightarrow{\frac{1}{3}R_4 \to R_4} \left(\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right) \xrightarrow{\substack{-4R_4+R_1 \to R_1 \\ R_4+R_2 \to R_2}} \left(\begin{array}{ccccc} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right) \xrightarrow[R_3 \leftrightarrow R_4]{} \left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \begin{aligned} x_1 + 2x_3 &= 0 \\ x_2 - 3x_3 &= 4 \\ x_4 &= -2 \end{aligned} \quad \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2x_3 \\ 3x_3+4 \\ x_3 \\ -2 \end{pmatrix} = x_3 \begin{pmatrix} -2 \\ 3 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \\ 0 \\ -2 \end{pmatrix}$$

$\operatorname{rank} A = 3$

2. (25 points) Let $L(\mathbf{x}) = A\mathbf{x}$ a linear transformation w/ $A$ def'd by:

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 8 \\ 2 & 7 & 12 \end{bmatrix}$$

Calculate the inverse of $A$ if it exists. What is one condition that will ensure invertibility of a square matrix?

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 8 & 0 & 1 & 0 \\ 2 & 7 & 12 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R3-2\times R1]{R2-R1} \left(\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & -1 & 1 & 0 \\ 0 & 1 & 6 & -2 & 0 & 1 \end{array}\right) \xrightarrow{R3-R2} \left(\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow[R1-3\times R3]{R2-5\times R3} \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 4 & 3 & -3 \\ 0 & 1 & 0 & 4 & 6 & -5 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right) \xrightarrow{R1-3\times R2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -8 & -15 & 12 \\ 0 & 1 & 0 & 4 & 6 & -5 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right)$$

$$\boxed{A^{-1} = \begin{pmatrix} -8 & -15 & 12 \\ 4 & 6 & -5 \\ 1 & -1 & 1 \end{pmatrix}}$$

Conditions for an $n \times n$ matrix to be invertible. (pick one)

- $\det A \neq 0$.
- $\operatorname{rk} A = n$.
- $\operatorname{rref}[A \mid I_n]$ is of the form $[I_n \mid B]$

3. (25 points)

(a) Give an example of a $3 \times 2$ matrix $A$ and a vector $\mathbf{b}$ such that the system $A\mathbf{x} = \mathbf{b}$ has a unique solution. Why doesn't this contradict the invertibility criteria of matrices?

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ $b = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Then $x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a unique solution.

A is a non-square mtx, so it doesn't contradict to the criteria.

(b) Does the matrix

$$A = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

represents a rotation? Explain very briefly.

$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix}$ = composition of scaling by $\frac{1}{\sqrt{2}}$ and rotating by $\frac{\pi}{4}$

(c) Is the function

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 1 \end{bmatrix}$$

a linear transformation? Explain.

No, $T\begin{bmatrix} x \\ y \end{bmatrix} + T\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ 1 \end{bmatrix} + \begin{bmatrix} y' \\ 1 \end{bmatrix} = \begin{bmatrix} y+y' \\ 2 \end{bmatrix}$

$\neq$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x' \\ y' \end{bmatrix}\right) = T\left(\begin{bmatrix} x+x' \\ y+y' \end{bmatrix}\right) = \begin{bmatrix} y+y' \\ 1 \end{bmatrix}$

4. (25 points) Let $T(\mathbf{x}) = A\mathbf{x}$ a linear transformation w/ $A$ def'd by:

$$A = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 4 \end{bmatrix}$$

(a) Find a spanning set for the Kernel of $A$

(b) Find a spanning set for the Image of $A$ that consists of only 3 vectors. (e.g. one of the vectors may be a linear combination of the others and hence omittable in the span.)

(c) Are the columns of $A$ linearly independent? Explain Clearly.

a) By #1, $\mathrm{rref}(A) = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Since $\ker A = \ker(\mathrm{rref}(A))$

and $\underset{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}{\mathbf{x}} \in \ker(\mathrm{rref}(A))$ implies that $\begin{cases} x_1 + 2x_3 = 0 \\ x_2 - 3x_3 = 0 \\ x_4 = 0 \end{cases}$,

$$\ker A = \left\{ x_3 \begin{pmatrix} -2 \\ 3 \\ 1 \\ 0 \end{pmatrix} \;\middle|\; \forall x_3 \in \mathbb{R} \right\}.$$

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b) Let $c_i$ be the columns of $A$ for $1 \leq i \leq 4$

Suppose that $x_1c_1 + x_2c_2 + x_3c_3 + x_4c_4 = \vec{0}$. Then we have

$x_1 + 2x_3 = 0$, $x_2 - 3x_3 = 0$ and $x_4 = 0$ from a). So $x_3(-2c_1 + 3c_2 + c_3) = 0$

for $\forall x_3 \in \mathbb{R}$. $\Rightarrow$ $c_3 = 2c_1 - 3c_2$

$$\mathrm{Im}\, A = \left\langle \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 4 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \\ 8 \\ 4 \end{pmatrix} \right\rangle$$



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c) No, $c_3$ is a combination of $c_1$ and $c_2$