Math 33A
Linear Algebra and Applications

Midterm 2

Instructions: You have 24 hours to complete this exam. There are 7 questions, worth a total of 100 points. This test is closed book and closed notes. No calculator is allowed. This document is the template where you need to provide your answers. Please print or download this document, complete it in the space provid d, show your work in the space provided, clearly box your final answer, and upload a pdf version of this document with your solutions. Do not upload a different document, and do not upload loose paper sheets. Do not forget to write your name, section (if you do not know your section, please write the name of your TA), and UID in the space below. Failure to comply with any of these instructions may have repercussions in your final grade.

Question	Points	Score
1	10	
2	15	
3	15	
4	15	
5	15	
6	15	
7	15	
Total:	100	

Problem 1. *10pts.*

Determine whether the following statements are true or false.

(a) If A and B are symmetric $n \times n$ matrices, then $ABBA$ must be symmetric as well.

$\boxed{T}$

$(ABBA)^T = A^T(ABB)^T = A^T B^T (AB)^T$

$= A^T B^T B^T A^T$ $\quad A = A^T \quad B = B^T \quad \rightarrow ABBA$

(b) If A is an invertible matrix such that $A^{-1} = A$, then A must be orthogonal.

$\boxed{F}$

$A = \begin{bmatrix} 1 & 5 \\ 0 & -1 \end{bmatrix}$ not orthogonal

$A^{-1} = \frac{1}{-1}\begin{bmatrix} -1 & -5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 0 & -1 \end{bmatrix} = A$

(c) If V is a subspace of $\mathbb{R}^n$ and $\vec{x}$ is a vector in $\mathbb{R}^n$, then the inequality $\vec{x} \cdot (\text{proj}_V \vec{x}) \geq 0$ must hold.

$\boxed{T}$

Two cases: Angle between x and V:

$< 90°$: $\Rightarrow \vec{x}$ and $\text{proj}_V \vec{x}$ point in same direction $\Rightarrow$ positive

$> 90°$: $\vec{x}$ and $\text{proj}_V x$ point in same direction $\Rightarrow$ positive

(d) If matrix B is obtained by swapping two rows of an $n \times n$ matrix A, then the equation $\det(B) = -\det(A)$ must hold.

$\boxed{T}$

(e) There exist real invertible 3×3 matrices A and S such that $S^T AS = -A$.

$\boxed{F}$

$A = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$\det(S^T AS) = \det(-A)$

$\det(S^T)\det(S)\det(A) = (-1)^3\det(A)$

$[\det(S)]^2 \det(A) = -\det(A)$

because $[\det(S)]^2 \geq 0 \Rightarrow$ equation is false

Problem 2. *15pts.*

Consider the vectors

$$\vec{u_1} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad \vec{u_2} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \quad \vec{u_3} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

in $\mathbb{R}^4$. Is there a vector $\vec{u_4}$ in $\mathbb{R}^4$ such that $\mathfrak{B} = \{\vec{u_1}, \vec{u_2}, \vec{u_3}, \vec{u_4}\}$ is an orthonormal basis? If so, how many such vectors are there?

Let $\vec{V_4} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}$ then: $\vec{V_4}^{\parallel} = (\vec{u_1} \cdot \vec{V_4})\vec{u_1} + (\vec{u_2} \cdot \vec{V_4})\vec{u_2} + (\vec{u_3} \cdot \vec{V_4})\vec{u_3}$

$$= \frac{5}{2}\vec{u_1} - \frac{1}{2}\vec{u_2} - \frac{1}{2}\vec{u_3}$$

$$= \begin{bmatrix} 5/4 \\ 5/4 \\ 5/4 \\ 5/4 \end{bmatrix} + \begin{bmatrix} -1/4 \\ -1/4 \\ 1/4 \\ 1/4 \end{bmatrix} + \begin{bmatrix} -1/4 \\ 1/4 \\ -1/4 \\ 1/4 \end{bmatrix}$$

$$= \begin{bmatrix} 3/4 \\ 5/4 \\ 5/4 \\ 7/4 \end{bmatrix}$$

$$\vec{V_4}^{\perp} = \vec{V_4} - \vec{V_4}^{\parallel} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3/4 \\ 5/4 \\ 5/4 \\ 7/4 \end{bmatrix} = \begin{bmatrix} 1/4 \\ -1/4 \\ -1/4 \\ 1/4 \end{bmatrix}$$

$$\vec{u_4} = \frac{1}{\|V_4^{\perp}\|} V_4^{\perp} = 2\begin{bmatrix} 1/4 \\ -1/4 \\ -1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$\Rightarrow$ There is a vector $\vec{u_4}$ in $\mathbb{R}^4$ such that $\mathfrak{B}$ is an orthonormal basis

Only 2 possible vectors that has length 1 and $\perp$ to subspace (90° and 180° of angle between)

There are 2 such vectors; $\begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$ and $\begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$

One is $\vec{u_4}$ and the other is $-\vec{u_4}$ (pointing in the opposite direction of the $\hat{u_4}$ that are found)

Problem 3. *15pts.*

Find the QR factorization of the following matrix.

$$M = \frac{1}{2}\begin{bmatrix}1&1&1\\1&-1&-1\\1&-1&1\\1&1&-1\end{bmatrix}\begin{bmatrix}2&3&5\\0&-4&6\\0&0&7\end{bmatrix}.$$

Let $Q = \frac{1}{2}\begin{bmatrix}1&1&1\\1&-1&-1\\1&-1&1\\1&1&-1\end{bmatrix} = \begin{bmatrix}\vec{u}_1 & \vec{u}_2 & \vec{u}_3\end{bmatrix}$ We can verify that Q's column vectors are orthonormal: $||\vec{u}_1|| = \sqrt{(\frac{1}{2})^2+(\frac{1}{2})^2+(\frac{1}{2})^2+(\frac{1}{2})^2} = \sqrt{1} = 1$

$||\vec{u}_2|| = \sqrt{(\frac{1}{2})^2+(-\frac{1}{2})^2+(-\frac{1}{2})^2+(\frac{1}{2})^2} = \sqrt{1} = 1$ $\quad ||\vec{u}_3|| = \sqrt{(\frac{1}{2})^2+(-\frac{1}{2})^2+(\frac{1}{2})^2+(-\frac{1}{2})^2} = 1$

$$\left.\begin{aligned}\vec{u}_1\cdot\vec{u}_2 &= \tfrac{1}{4}-\tfrac{1}{4}-\tfrac{1}{4}+\tfrac{1}{4} = 0\\ \vec{u}_1\cdot\vec{u}_3 &= \tfrac{1}{4}-\tfrac{1}{4}+\tfrac{1}{4}-\tfrac{1}{4} = 0\\ \vec{u}_2\cdot\vec{u}_3 &= \tfrac{1}{4}+\tfrac{1}{4}-\tfrac{1}{4}-\tfrac{1}{4} = 0\end{aligned}\right\}\text{ Vectors are orthogonal to each other}$$

$\therefore$ Q is an orthogonal matrix

R needs to have ~~positive diagonal~~ entries:

From M, we can extrapolate that $\vec{v}_2 = 3\vec{u}_1 - 4\vec{u}_2$. We need the scalar of $\vec{u}_2$ to be positive, so we set $\vec{u}_2{}' = -\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix} = \begin{bmatrix}-1\\1\\1\\-1\end{bmatrix}$.

We now have $\vec{v}_2 = 3\vec{u}_1 + 4\vec{u}_2{}'$. We check $\vec{v}_3 = 5\vec{u}_1 + 6\vec{u}_2 + 7\vec{u}_3$

$= 5\vec{u}_1 - 6\vec{u}_2{}' + 7\vec{u}_3$

$\therefore$ Our QR factorization is:

$$M = \underbrace{\begin{bmatrix}\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\\ \frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\\ \frac{1}{2}&\frac{1}{2}&\frac{1}{2}\\ \frac{1}{2}&-\frac{1}{2}&-\frac{1}{2}\end{bmatrix}}_{Q}\underbrace{\begin{bmatrix}2&3&5\\0&4&-6\\0&0&7\end{bmatrix}}_{R}$$

Problem 4. *15pts.*

Find an orthogonal matrix of the form

$$\begin{bmatrix} 2/3 & 1/\sqrt{2} & a \\ 2/3 & -1/\sqrt{2} & b \\ 1/3 & 0 & c \end{bmatrix}.$$

$\vec{u}_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ Grand - Shmilt Process!

let $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\vec{v}_3^{\,\parallel} = (\vec{u}_1 \cdot \vec{v}_3) \cdot \vec{u}_1 + (\vec{u}_2 \cdot \vec{v}_3) \cdot \vec{u}_2 = \frac{5}{3}\vec{u}_1 = \begin{bmatrix} \frac{10}{9} \\ \frac{10}{9} \\ \frac{5}{9} \end{bmatrix}$$

$$\vec{v}_3^{\,\perp} = \vec{v}_3 - v_3^{\parallel} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{10}{9} \\ \frac{10}{9} \\ \frac{5}{9} \end{bmatrix} = \begin{bmatrix} -\frac{1}{9} \\ -\frac{1}{9} \\ \frac{4}{9} \end{bmatrix}$$

$$\vec{u}_3 = \frac{1}{\|v_3^{\perp}\|} v_3^{\perp} = \frac{1}{\sqrt{\frac{18}{81}}} \begin{bmatrix} -\frac{1}{9} \\ -\frac{1}{9} \\ \frac{4}{9} \end{bmatrix} = \frac{9}{\sqrt{18}} \begin{bmatrix} -\frac{1}{9} \\ -\frac{1}{9} \\ \frac{4}{9} \end{bmatrix} = \frac{3}{\sqrt{2}} \begin{bmatrix} -\frac{1}{9} \\ -\frac{1}{9} \\ \frac{4}{9} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-1}{3\sqrt{2}} \\ \frac{-1}{3\sqrt{2}} \\ \frac{4}{3\sqrt{2}} \end{bmatrix}$$

∴ An orthogonal matrix matches the form is

$$\cancel{\begin{bmatrix} 2/3 & 1/\sqrt{2} & -1/3\sqrt{2} \\ 2/3 & -1/\sqrt{2} & \frac{-1}{3\sqrt{2}} \\ 1/3 & 0 & \frac{4}{3\sqrt{2}} \end{bmatrix}} \quad \begin{bmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{-1}{3\sqrt{2}} \\ \frac{2}{3} & \frac{-1}{\sqrt{2}} & \frac{-1}{3\sqrt{2}} \\ \frac{1}{3} & 0 & \frac{4}{3\sqrt{2}} \end{bmatrix}$$

Problem 5. *15pts.*

Find the least-squares solution $\vec{x}^*$ of the system $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 3 & 2 \\ 5 & 3 \\ 4 & 5 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 5 \\ 9 \\ 2 \end{bmatrix}.$$

Draw a sketch showing the vector $\vec{b}$, the image of A, the vector $A\vec{x}^*$, and the vector $\vec{b} - A\vec{x}^*$.

$$\vec{x}^* = (A^T A)^{-1} A^T \vec{b}$$

$$= \left(\begin{bmatrix} 3 & 5 & 4 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & 3 \\ 4 & 5 \end{bmatrix} \right)^{-1} \begin{bmatrix} 3 & 5 & 4 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ 9 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 50 & 41 \\ 41 & 38 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 68 \\ 47 \end{bmatrix}$$

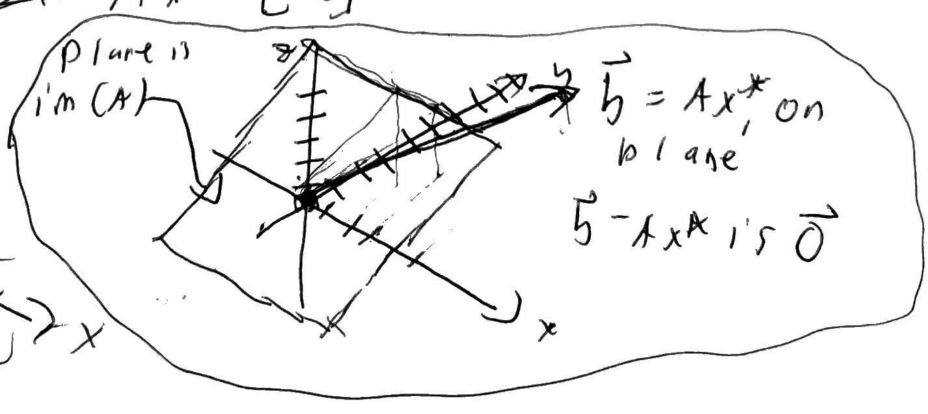
$$= \frac{1}{1900 - 1681} \begin{bmatrix} 38 & -41 \\ -41 & 50 \end{bmatrix} \cdot \begin{bmatrix} 68 \\ 47 \end{bmatrix}$$

$$= \frac{1}{219} \begin{bmatrix} 38 & -41 \\ -41 & 50 \end{bmatrix} \begin{bmatrix} 68 \\ 47 \end{bmatrix}$$

$$= \frac{1}{219} \begin{bmatrix} 2584 - 1927 \\ -2788 + 2350 \end{bmatrix} = \frac{1}{219} \begin{bmatrix} 657 \\ -438 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Wrong sketch

$$Ax^* = \begin{bmatrix} 5 \\ 9 \\ 2 \end{bmatrix} = \vec{b} \Rightarrow \vec{b} - A\vec{x}^* = 0$$



Problem 6. *15pts.*

Consider a 4×4 matrix A with rows $\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}$. If $\det(A) = 8$, find

$$\det \begin{bmatrix} 6\vec{v_1} + 2\vec{v_4} \\ \vec{v_2} \\ \vec{v_3} \\ 3\vec{v_1} + \vec{v_4} \end{bmatrix}.$$

$$\det \begin{bmatrix} 6\vec{v_1} + 2\vec{v_4} \\ \vec{v_2} \\ \vec{v_3} \\ 3\vec{v_1} + \vec{v_4} \end{bmatrix} = \det \begin{bmatrix} 6\vec{v_1} \\ \vec{v_2} \\ \vec{v_3} \\ 3\vec{v_1} \end{bmatrix}$$

Rows 1 and 4 are linearly dependent because row 1 is a multiple of row 4 (by a factor of 2)

Because 2 row vectors are linearly dependent, the determinant is

0

Problem 7. *15pts.*

Find the classical adjoint of the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

and use the result to find A^{-1}.

$$\cancel{\operatorname{adj} A = \begin{bmatrix} 1 & & \\ & -1 & \\ & & 1 \end{bmatrix}}$$

$$\operatorname{adj}(A) = \begin{bmatrix} \det(A_{11}) & -\det(A_{21}) & \det(A_{31}) \\ -\det(A_{12}) & \det(A_{22}) & -\det(A_{32}) \\ \det(A_{13}) & -\det(A_{23}) & \det(A_{33}) \end{bmatrix}$$

$$\cancel{\operatorname{adj}(A) = \begin{bmatrix} \frac{\det(A_{11})}{\det(A)} & -\frac{\det(A_{21})}{\det(A)} & \frac{\det(A_{31})}{\det(A)} \\ \frac{\det(A_{12})}{\det(A)} & \frac{\det(A_{22})}{\det(A)} & \frac{\det(A_{32})}{\det A} \\ \frac{\det(A_{13})}{\det(A)} & -\frac{\det(A_{21})}{\det(A)} & \frac{\det(A_{33})}{\det(A)} \end{bmatrix}}$$

$$\operatorname{adj}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$\det(A) = 1\det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \det\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} = 1 - 2 = -1$$

$$A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$$