22W-MATH-33A-LEC-2 Midterm 2

TOTAL POINTS

100 / 100

QUESTION 1

Problem 1 10 pts

1.1 (a) 2 / 2

✓ **+ 2 pts Correct**

+ 0 pts Not correct

1.2 (b) 2 / 2

✓ **+ 2 pts Correct**

+ 0 pts Not correct

1.3 (c) 2 / 2

✓ **+ 2 pts Correct**

+ 0 pts Not correct

1.4 (d) 2 / 2

✓ **+ 2 pts Correct**

+ 0 pts Not correct

1.5 (e) 2 / 2

✓ **+ 2 pts Correct**

+ 0 pts Not correct

QUESTION 2

2 Problem 2 15 / 15

✓ **+ 15 pts Correct**

+ 10 pts Found only one of the two possible vectors

+ 12 pts Gave the correct vectors, and an extra incorrect one

+ 12 pts Found two vectors, but didn't normalize the vectors

+ 9 pts Claimed all orthogonal vectors work, didn't normalize.

+ 13 pts Correct, but poor explanation/justification

+ 5 pts Showed $$u_1,u_2,u_3$$ are orthogonal, but little further progress or justification.

QUESTION 3

3 Problem 3 15 / 15

✓ **- 0 pts Correct**

- 2 pts minor computational/arithmetic errors

- 5 pts Attempted to use Gram-Schmidt but did so with significant conceptual errors

- 5 pts Attempted without using Gram-Schmidt but didn't check that Q is an orthogonal matrix

- 2 pts Attempted without using Gram-Schmidt and checked that Q has orthogonal columns but didn't check that the columns have length 1.

- 5 pts Attempted without using Gram-Schmidt but didn't try to change R to have positive diagonal entries

- 15 pts Incorrect / no answer

QUESTION 4

4 Problem 4 15 / 15

✓ **+ 15 pts Correct**

+ 7.5 pts (a, b, c) is orthogonal to other columns

+ 7.5 pts (a, b, c) has length 1

+ 3.75 pts Attempted orthogonality to other columns but made an error somewhere

+ 3.75 pts Attempted normalization of vector but made an error somewhere

QUESTION 5

5 Problem 5 15 / 15

Set up equation to find $$x^*$$

✓ **+ 5 pts Multiply on the left by $$A^T$$ on both sides**

- 2 pts Computational mistake when multiplying

Correctly solve for least squares solution

$$x^*$$

✓ **+ 5 pts** **Solve equation correctly (e.g. use inverse of 2 by 2, or row reduce)**

- 2 pts Computational mistake

Sketch

✓ **+ 5 pts** **Sketch the vectors $$b, Ax^*, b-Ax^*$$, and the subspace $$Im(A)$$**

- 2 pts At least one problem with sketch (e.g. something missing, drew distinct vectors for $$Ax^*$$ and $$b$$, vectors are not labeled and not otherwise clarified, did not draw $$Im(A)$$ to be two dimensional/did not draw $$Im(A)$$ to contain $$b=Ax^*$$, etc.).

Note that just drawing two vectors that span $$Im(A)$$ is not the same as drawing $$Im(A)$$; you should indicate in some way that the image is the span of those two vectors or that the image is two dimensional.

- 1 pts At least two problems with sketch

- 1 pts At least three problems with sketch

- 1 pts Four problems with sketch (i.e. sketch did not include any of the four things it was supposed to include)

+ 2 pts Did not draw sketch but correctly noted relevant information for the sketch (e.g. that $$b-Ax^*=0$$ and that $$Im(A)$$ is two dimensional, spanned by the two columns of $$A$$)

QUESTION 6

6 Problem 6 15 / 15

✓ **- 0 pts** **Correct**

- 3 pts Reduced incorrectly by pulling out the inverse of the term, e.g. got a factor of $$\frac{1}{2}$$ instead of $$2$$

- 5 pts Some justification shown, but not enough/not clear.

- 9 pts Did not give proper explanation or justification

- 7 pts Did multiple row operations at once, which is not valid and does not give the correct reduction

- 5 pts Missing term(s) in the expansion of $$\text{det}\begin{bmatrix}
6v_1+2v_4\\
v_2\\
v_3\\
3v_1+v_4
\end{bmatrix}$$

- 5 pts Properly expanded $$\text{det}\begin{bmatrix}
6v_1+2v_4\\
v_2\\
v_3\\
3v_1+v_4
\end{bmatrix}$$ into $$\text{det}\begin{bmatrix}
6v_1\\
v_2\\
v_3\\
3v_1
\end{bmatrix}+\text{det}\begin{bmatrix}
6v_1\\
v_2\\
v_3\\
v_4
\end{bmatrix}+\text{det}\begin{bmatrix}
2v_4\\
v_2\\
v_3\\
3v_1
\end{bmatrix}+\text{det}\begin{bmatrix}
2v_4\\
v_2\\
v_3\\
v_4
\end{bmatrix}$$, but did not properly compute their values

- 1 pts Very minor error

- 2 pts Minor error

- 15 pts No points

QUESTION 7

7 Problem 7 15 / 15

Computing classical adjoint

✓ **+ 7 pts Computed the cofactor matrix correctly**

- 2 pts Did not remember to add a sign of $$(-1)^{i+j}$$ for the i,j cofactor, or other sign mistake

✓ **+ 2 pts Took the transpose of the cofactor matrix to get the adjugate matrix**

Computing $$A^{-1}$$

✓ **+ 4 pts Correctly computed det(A)**

- 2 pts Mixed up signs when computing det(A)

✓ **+ 2 pts Correctly computed $$A^{-1}$$ (relative to the previously computed values for adj(A) and det(A))**

- 1 pts General computational/notational/other small but meaningful mistake

+ 0 pts No solution given

Problem 1. *10pts.*

Determine whether the following statements are true or false.

(a) If A and B are symmetric $n \times n$ matrices, then $ABBA$ must be symmetric as well.

T

$(ABBA)^T = ((AB)(BA))^T = (BA)^T(AB)^T$
$= (A^TB^T)(B^TA^T)$
$= ABBA$

(b) If A is an invertible matrix such that $A^{-1} = A$, then A must be orthogonal.

F

$\det(A^{-1}A) = \det(I_n) = 1$
$= \det(A)^2$
Let $\det(A) = -1$

2x2:
A A^{-1}
$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -d & b \\ c & -a \end{bmatrix}$
counterexample:
$A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$

(c) If V is a subspace of $\mathbb{R}^n$ and $\vec{x}$ is a vector in $\mathbb{R}^n$, then the inequality $\vec{x} \cdot (\text{proj}_V \vec{x}) \geq 0$ must hold.

T

(d) If matrix B is obtained by swapping two rows of an $n \times n$ matrix A, then the equation $\det(B) = -\det(A)$ must hold.

T

(e) There exist real invertible 3×3 matrices A and S such that $S^TAS = -A$.

F

$\det(S^TAS) = \det(-A)$
$= \det(S^T)\det(A)\det(S)$
$= \det(S)^2 \det(A) = -\det(A)$

3 is odd
$(-1)^3 = -1$

$\det(S)^2 = -1$
impossible
for real
invertible 3x3
matrices

Problem 2. *15pts.*

Consider the vectors

$$\vec{u_1} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad \vec{u_2} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \quad \vec{u_3} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

in $\mathbb{R}^4$. Is there a vector $\vec{u_4}$ in $\mathbb{R}^4$ such that $\mathfrak{B} = \{\vec{u_1}, \vec{u_2}, \vec{u_3}, \vec{u_4}\}$ is an orthonormal basis? If so, how many such vectors are there?

3 orthonormal vectors span a 3-dimensional subspace in $\mathbb{R}^4$. We can add an orthonormal vector to $\vec{u}_1, \vec{u}_2, \vec{u}_3$ to form an orthonormal basis $\mathcal{B} = \{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ spanning 4 dimensions.

Example: Consider the unit vector $\begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$. Let $\vec{u}_4$ be this vector.

$$\vec{u}_1 \cdot \vec{u}_4 = \left(\tfrac{1}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2 = 0$$

$$\vec{u}_2 \cdot \vec{u}_4 = \left(\tfrac{1}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2 = 0$$

$$\vec{u}_3 \cdot \vec{u}_4 = \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2 = 0$$

$$\sqrt{\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2} = 1$$

Clearly, there is indeed a vector $\vec{u}_4$ in $\mathbb{R}^4$ such that $\mathcal{B} = \{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ is an orthonormal basis.

There are exactly 2 such vectors.

$\begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$ is one. If this is orthogonal to $\vec{u}_1, \vec{u}_2, \vec{u}_3$ then

$-\begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$ is another, interpreting dot product geometrically.

There are no more such vectors!

$$\operatorname{im}\left(\begin{bmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | \end{bmatrix}\right)^{\perp} = \ker\left(\begin{bmatrix} - \vec{u}_1 - \\ - \vec{u}_2 - \\ - \vec{u}_3 - \end{bmatrix}\right)$$

$$\begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \end{bmatrix} \xrightarrow{\begin{bmatrix} R_1 + R_3 \\ R_1 - R_3 \\ R_1 - R_2 \end{bmatrix}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 - R_3} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\operatorname{rank}\left(\begin{bmatrix} - \vec{u}_1 - \\ - \vec{u}_2 - \\ - \vec{u}_3 - \end{bmatrix}\right) = 3$$

Rank-Nullity:

$$\dim \ker\left(\begin{bmatrix} - \vec{u}_1 - \\ - \vec{u}_2 - \\ - \vec{u}_3 - \end{bmatrix}\right) = 1$$

1-dimensional: only 2 vectors of magnitude 1

Problem 3. *15pts.*

Find the QR factorization of the following matrix.

$$M = \underbrace{\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}}\begin{bmatrix} 2 & 3 & 5 \\ 0 & -4 & 6 \\ 0 & 0 & 7 \end{bmatrix}.$$

close to a QR factorization, but invalid because of diagonal

problematic entry is negative (R must have positive diagonal.)

$\frac{1}{2}\sqrt{1+1+1+1} = 1$

Note this is orthogonal.

Note $\frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$

is still orthogonal.

We can try multiplying the column $\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$ that interacts with this second row by -1: $\begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$.

in matrix multiplication

We have

$$M = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 5 \\ 0 & -4 & 6 \\ 0 & 0 & 7 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & -6 \\ 0 & 0 & 7 \end{bmatrix}$$

Note that we have to multiply the row $[0\ -4\ 6]$ by -1 as well because it interacts with column $\begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$ in matrix multiplication.

So, a valid QR factorization is

$$\boxed{M = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & -6 \\ 0 & 0 & 7 \end{bmatrix}}$$

where $Q = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$

$R = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & -6 \\ 0 & 0 & 7 \end{bmatrix}$

Problem 4. *15pts.*

Find an orthogonal matrix of the form

$$\begin{array}{c} \begin{array}{ccc} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{array} \\ \begin{bmatrix} 2/3 & 1/\sqrt{2} & a \\ 2/3 & -1/\sqrt{2} & b \\ 1/3 & 0 & c \end{bmatrix} \end{array}.$$

$$\vec{u}_1 \cdot \vec{u}_3 = \frac{2}{3}a + \frac{2}{3}b + \frac{1}{3}c = 0$$

$$\vec{u}_2 \cdot \vec{u}_3 = \frac{1}{\sqrt{2}}a - \frac{1}{\sqrt{2}}b + 0 = 0$$

$$a = b$$

$$\frac{4}{3}a + \frac{1}{3}c = 0$$

$$4a + c = 0$$

$$c = -4a$$

$$\|\vec{u}_3\| = \sqrt{a^2 + b^2 + c^2} = 1$$

$$= \sqrt{a^2 + a^2 + (-4a)^2}$$

$$= \sqrt{18a^2}$$

$$= 3\sqrt{2}\,|a| = 1$$

Between $-\frac{1}{3\sqrt{2}}$ and $\frac{1}{3\sqrt{2}}$, let $a = \frac{1}{3\sqrt{2}}$ arbitrarily.

$$b = \frac{1}{3\sqrt{2}}$$

$$c = -\frac{4}{3\sqrt{2}}$$

$$\boxed{\begin{bmatrix} 2/3 & 1/\sqrt{2} & 1/3\sqrt{2} \\ 2/3 & -1/\sqrt{2} & 1/3\sqrt{2} \\ 1/3 & 0 & -4/3\sqrt{2} \end{bmatrix}}$$

Problem 5. *15pts.*

Find the least-squares solution $\vec{x}^*$ of the system $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 3 & 2 \\ 5 & 3 \\ 4 & 5 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 5 \\ 9 \\ 2 \end{bmatrix}.$$

Draw a sketch showing the vector $\vec{b}$, the image of A, the vector $A\vec{x}^*$, and the vector $\vec{b} - A\vec{x}^*$.

$A^TA\vec{x}^* = A^T\vec{b}$ (normal equations)

$$A^T = \begin{bmatrix} 3 & 5 & 4 \\ 2 & 3 & 5 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 3 & 5 & 4 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 50 & 41 \\ 41 & 38 \end{bmatrix}$$

$$\begin{bmatrix} 50 & 41 \\ 41 & 38 \end{bmatrix} \vec{x}^* = \begin{bmatrix} 3 & 5 & 4 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ 9 \\ 2 \end{bmatrix} = \begin{bmatrix} 68 \\ 47 \end{bmatrix}$$

$3 \cdot 5 + 5 \cdot 9 + 4 \cdot 2 = 68$

$2 \cdot 5 + 3 \cdot 9 \cdot 5 \cdot 2 = 47$

$3^2 + 5^2 + 4^2 = 50$

$2^2 + 3^2 + 5^2 = 38$

$3 \cdot 2 + 5 \cdot 3 + 4 \cdot 5 = 41$

$50x_1 + 41x_2 = 68$

$41x_1 + 38x_2 = 47$

By inspection,

$x_1 = 3$

$x_2 = -2$

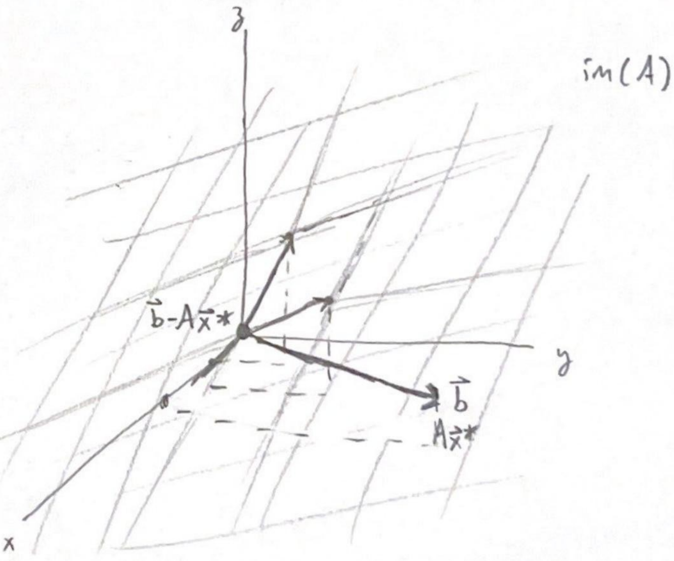
$(68 = 50 + 2(50 - 41))$

$(47 = 41 + 2(41 - 38))$

$$\boxed{\vec{x}^* = \begin{bmatrix} 3 \\ -2 \end{bmatrix}}$$

$$A\vec{x}^* = \begin{bmatrix} 3 & 2 \\ 5 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ 2 \end{bmatrix}$$

$$\vec{b} - A\vec{x}^* = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



$\vec{b}$ is in the image of A,

The projection of $\vec{b}$ is equal to $\vec{b}$.

$\vec{b} - A\vec{x}^*$ is the zero vector.

Problem 6. *15pts.*

Consider a 4×4 matrix A with rows $\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}$. If $\det(A) = 8$, find

$$\det \begin{bmatrix} 6\vec{v_1} + 2\vec{v_4} \\ \vec{v_2} \\ \vec{v_3} \\ 3\vec{v_1} + \vec{v_4} \end{bmatrix}.$$

$$\det \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_4 \end{bmatrix} = 8$$

$$= \det \begin{bmatrix} 6\vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \\ 3\vec{v}_1 + \vec{v}_4 \end{bmatrix} + \det \begin{bmatrix} 2\vec{v}_4 \\ \vec{v}_2 \\ \vec{v}_3 \\ 3\vec{v}_1 + \vec{v}_4 \end{bmatrix}$$

$$= \left(\det \begin{bmatrix} 6\vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \\ 3\vec{v}_1 \end{bmatrix} + \det \begin{bmatrix} 6\vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_4 \end{bmatrix} \right) + \left(\det \begin{bmatrix} 2\vec{v}_4 \\ \vec{v}_2 \\ \vec{v}_3 \\ 3\vec{v}_1 \end{bmatrix} + \det \begin{bmatrix} 2\vec{v}_4 \\ \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_4 \end{bmatrix} \right)$$

$$= 2\det \begin{bmatrix} 3\vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \\ 3\vec{v}_1 \end{bmatrix} + 6\det \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_4 \end{bmatrix} + 2\det \begin{bmatrix} \vec{v}_4 \\ \vec{v}_2 \\ \vec{v}_3 \\ 3\vec{v}_1 \end{bmatrix} + 2\det \begin{bmatrix} \vec{v}_4 \\ \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_4 \end{bmatrix}$$

$$= 0 + 6 \cdot 8 + 6\det \begin{bmatrix} \vec{v}_4 \\ \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_1 \end{bmatrix} + 0$$

$$= 48 - 6\det \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_4 \end{bmatrix}$$

$$= 48 - 6 \cdot 8$$

$$= 48 - 48$$

$$= 0$$

$$\boxed{\det \begin{bmatrix} 6\vec{v}_1 + 2\vec{v}_4 \\ \vec{v}_2 \\ \vec{v}_3 \\ 3\vec{v}_1 + \vec{v}_4 \end{bmatrix} = 0}$$

Problem 7. *15pts.*
Find the classical adjoint of the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

$\det(A) = 1(1-0) - 0 + 1(0-2)$
$= 1 - 2 = -1$

and use the result to find A^{-1}.

$$\operatorname{adj}(A) = \begin{bmatrix} \det(A_{11}) & -\det(A_{12}) & \det(A_{13}) \\ -\det(A_{21}) & \det(A_{22}) & -\det(A_{23}) \\ \det(A_{31}) & -\det(A_{32}) & \det(A_{33}) \end{bmatrix}^T$$

$$= \begin{bmatrix} 1-0 & -0+0 & 0-2 \\ -0+0 & 1-2 & -0+0 \\ 0-1 & -0+0 & 1-0 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$\boxed{\operatorname{adj}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix}}$$

$$A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)} = \frac{\operatorname{adj}(A)}{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$$

$$\boxed{A^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}}$$