Problem 1. *10pts.*

Determine whether the following statements are true or false. If the statement is true, write T in the box provided under the statement. If the statement is false, write F in the box provided under the statement. Do not write "true" or "false".

(a) If matrix A is in reduced row-echelon form, then at least one of the entries in each column must be 1.

F

Take $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$\text{rref}(A) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

(b) Consider an $n \times m$ matrix A. Can you transform rref(A) into A by a sequence of elementary row operations?

T

All elementary row operations to reach rref(A) are invertible

(c) The formula $(A^2)^{-1} = (A^{-1})^2$ holds for all invertible matrices A.

T

$(AA)^{-1} = A^{-1}A^{-1} = (A^{-1})^2$

(d) If $A^2 = I_2$, then matrix A must be either I_2 or $-I_2$.

F

Take $A = A^{-1}$

o r $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$

(e) If A and B are $n \times n$ matrices, and vector $\vec{v}$ is in the kernel of both A and B, then $\vec{v}$ must be in the kernel of matrix AB as well.

T

$T_1(\vec{x}) = A\vec{x}$ kernal of linear transformation contains $\vec{v}$

$T_2(\vec{x}) = AB\vec{x}$

$= A(B\vec{x})$

$= A\vec{w}$ where $\vec{w} = B\vec{x}$

then kernel of linear transformation must also contain $\vec{v}$

Problem 2. *15pts.*

Find all solutions of the linear system

$$\begin{aligned} x + 2y + 3z &= a \\ x + 3y + 8z &= b \\ x + 2y + 2z &= c \end{aligned}$$

where a, b, c are arbitrary constants.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 1 & 3 & 8 & b \\ 1 & 2 & 2 & c \end{array}\right] \begin{array}{l} \\ -(I) \\ -(I) \end{array} \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 1 & 5 & b-a \\ 0 & 0 & -1 & c-a \end{array}\right] \begin{array}{l} -2(II) \\ \\ \cdot -1 \end{array}$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & -7 & 3a-2b \\ 0 & 1 & 5 & b-a \\ 0 & 0 & 1 & a-c \end{array}\right] \begin{array}{l} +7(III) \\ -5(III) \\ \\ \end{array} \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 10a-7c-2b \\ 0 & 1 & 0 & b-6a+5c \\ 0 & 0 & 1 & a-c \end{array}\right]$$

$\therefore$ $x = 10a - 2b - 7c$

$\cancel{y = -6a + b + 5c}$

$y = -6a + b + 5c$

$z = a - c$

Problem 3. *15pts.*

Determine whether the vector

$$\begin{bmatrix} 1 \\ -1 \\ 8 \\ 2 \end{bmatrix}$$

is a linear combination of the vectors

$$\begin{bmatrix} 1 \\ 10 \\ 1 \\ 9 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 9 \\ 2 \\ 3 \\ 5 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 1 & 5 & 9 \\ -1 & 10 & 6 & 2 \\ 8 & 1 & 3 & 3 \\ 2 & 9 & 2 & 5 \end{bmatrix} \begin{matrix} \\ +(I) \\ -8(I) \\ -2(I) \end{matrix} \to \begin{bmatrix} 1 & 1 & 5 & 9 \\ 0 & 11 & 11 & 11 \\ 0 & -7 & -37 & -69 \\ 0 & 7 & -8 & -13 \end{bmatrix} \begin{matrix} \\ \div 11 \\ \\ \\ \end{matrix} \to \begin{bmatrix} 1 & 1 & 5 & 9 \\ 0 & 1 & 1 & 1 \\ 0 & -7 & -37 & -69 \\ 0 & 7 & -8 & -13 \end{bmatrix} \begin{matrix} -(II) \\ \\ +7(II) \\ -7(II) \end{matrix}$$

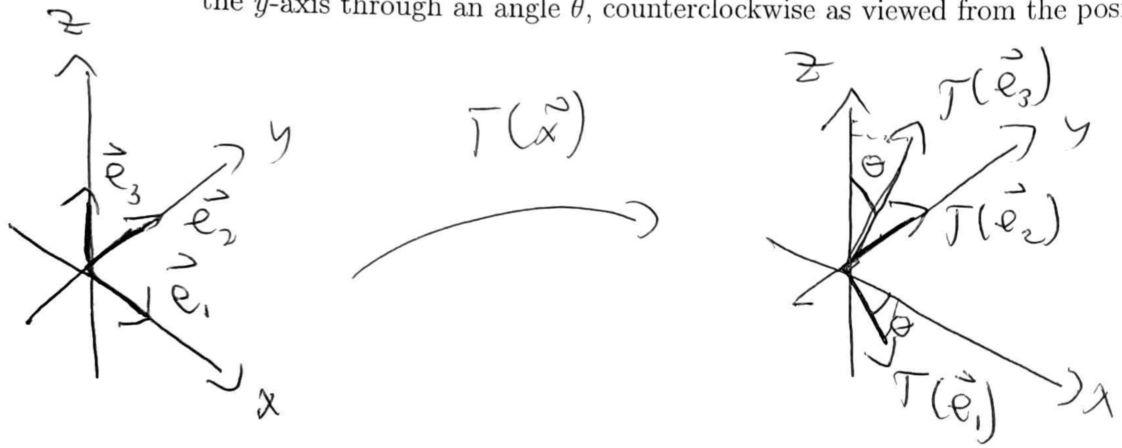
$$\begin{bmatrix} 1 & 0 & 4 & 8 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -30 & -62 \\ 0 & 0 & -15 & -20 \end{bmatrix} \begin{matrix} \\ \\ \div -30 \\ \\ \end{matrix} \begin{bmatrix} 1 & 0 & 4 & 8 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & \frac{31}{15} \\ 0 & 0 & -15 & -20 \end{bmatrix} \begin{matrix} -4(III) \\ -(III) \\ \\ +15(III) \end{matrix} \to \begin{bmatrix} 1 & 0 & 0 & (8-\frac{124}{15}) \\ 0 & 1 & 0 & -\frac{16}{15} \\ 0 & 0 & 1 & \frac{31}{15} \\ 0 & 0 & 0 & 11 \end{bmatrix} \begin{matrix} \\ \\ \\ \div 11 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & (8-\frac{124}{15}) \\ 0 & 1 & 0 & -\frac{16}{15} \\ 0 & 0 & 1 & \frac{31}{15} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} -(8-\frac{124}{15})(IV) \\ +\frac{16}{15}(IV) \\ -\frac{31}{15}(IV) \\ \\ \end{matrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Because the row-reduced echelon form of the matrix $A = \begin{bmatrix} 1 & 1 & 5 & 9 \\ -1 & 10 & 6 & 2 \\ 8 & 1 & 3 & 3 \\ 2 & 9 & 2 & 5 \end{bmatrix} = I_4$, the column vectors of A are linearly independent $\Rightarrow$ $\begin{bmatrix} 1 \\ -1 \\ 8 \\ 2 \end{bmatrix}$ is not a linear combination of the other 3 vectors.

Problem 4. *15pts.*

Find the matrix of the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ given by a rotation about the y-axis through an angle θ, counterclockwise as viewed from the positive y-axis.



$$T(\vec{e}_1) = \begin{bmatrix} \cos\theta \\ 0 \\ -\sin\theta \end{bmatrix} \quad T(\vec{e}_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad T(\vec{e}_3) = \begin{bmatrix} \sin\theta \\ 0 \\ \cos\theta \end{bmatrix}$$

$\therefore$ Matrix of linear transformation is:

$$\begin{bmatrix} | & | & | \\ T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \\ | & | & | \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

Problem 5. *15pts.*

Decide whether the matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

is invertible. If it is, find the inverse.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 3 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ -\mathrm{I} \\ -\mathrm{I} \end{array} \to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} -2(\mathrm{II}) \\ \\ +2(\mathrm{II}) \end{array}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 3 & -2 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & -3 & 2 & 1 \end{array}\right] \begin{array}{l} \\ \\ \div 2 \end{array} \to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 3 & -2 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 1 & \frac{1}{2} \end{array}\right] \begin{array}{l} +(\mathrm{III}) \\ -(\mathrm{III}) \\ \\ \end{array}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & -1 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{3}{2} & 1 & \frac{1}{2} \end{array}\right]$$

Left side $= I_3 \Rightarrow$ matrix is invertible with the inverse being

$$\begin{bmatrix} \frac{3}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{3}{2} & 1 & \frac{1}{2} \end{bmatrix}$$

Problem 6. *15pts.*

Describe, geometrically and algebraically, the image and kernel of the transformation in $\mathbb{R}^3$ given by the orthogonal projection onto the plane $x + 2y + 3z = 0$. In particular, find a basis of the image and a basis of the kernel.

Image

Image of the transformation of the orthogonal projection onto the plane is the plane itself, so ~~any~~ the span of any two linearly independent vectors ~~make up~~ is the image of the plane;

$z=0$: $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ $y=0$: $\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$ $\Rightarrow \text{img}(T(\vec{x})) = \text{Span}\left(\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}\right)$

Geometrically, it is the plane $x + 2y + 3z = 0$

Because $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$ are linearly independent, a basis for the image is $\left\{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}\right\}$

Kernal

The kernal of the transformation is the line orthogonal to the plane. Algebraically, $\text{Ker}(T) = \text{Span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$

The basis of the kernel is $\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}$

Problem 7. *15pts.*

Find a basis $\mathfrak{B}$ of $\mathbb{R}^2$ such that

$$\begin{bmatrix}1\\2\end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix}3\\5\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}3\\4\end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix}2\\3\end{bmatrix}.$$

Let basis $B = \{\vec{v}_1, \vec{v}_2\}$

$$\cancel{\text{Then } \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}_B = \begin{bmatrix}3\\5\end{bmatrix} \text{ and } \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix}_B = \begin{bmatrix}2\\3\end{bmatrix}}$$

Then $\begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix}\begin{bmatrix}3\\5\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix}$

$\Rightarrow 3v_1 + 5v_2 = \begin{bmatrix}1\\2\end{bmatrix}$ and $2v_1 + 3v_2 = \begin{bmatrix}3\\4\end{bmatrix}$

$$\Rightarrow \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix}\begin{bmatrix}3 & 2\\5 & 3\end{bmatrix} = \begin{bmatrix}1 & 3\\2 & 4\end{bmatrix}$$

$$\begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} = \begin{bmatrix}1 & 3\\2 & 4\end{bmatrix}\begin{bmatrix}3 & 2\\5 & 3\end{bmatrix}^{-1} = \begin{bmatrix}1 & 3\\2 & 4\end{bmatrix}\begin{bmatrix}-3 & 2\\5 & -3\end{bmatrix}$$

$$= \begin{bmatrix}12 & -7\\14 & -8\end{bmatrix}$$

$$\therefore \; B = \left\{ \begin{bmatrix}12\\14\end{bmatrix}, \begin{bmatrix}-7\\-8\end{bmatrix} \right\}$$