| Question | Points | Score |
|---|---|---|
| 1 | 5 | |
| 2 | 5 | |
| 3 | 7 | |
| 4 | 8 | |
| 5 | 8 | |
| 6 | 7 | |
| 7 | 8 | |
| 8 | 8 | |
| 9 | 8 | |
| 10 | 8 | |
| 11 | 7 | |
| 12 | 7 | |
| 13 | 7 | |
| 14 | 7 | |
| Total: | 100 | |

**Problem 1**. *5pts.*

Determine whether the following statements are true or false.

(a) If $A$ and $B$ are symmetric $n \times n$ matrices, then $ABBA$ must be symmetric as well.

T

$(ABBA)^T = A^T B^T B^T A^T = ABBA$

(b) The span of vectors $\vec{v_1}, \ldots, \vec{v_n}$ consists of all linear combinations of vectors $\vec{v_1}, \ldots, \vec{v_n}$.

T

(c) If two nonzero vectors are linearly dependent, then each of them is a scalar multiple of the other.

T

(d) If $A$ and $B$ are symmetric $n \times n$ matrices, then $AB$ must be symmetric as well.

F

$(AB)^T = B^T A^T = BA \neq AB$

(e) There exists a nonzero $4 \times 4$ matrix $A$ such that $\det(A) = \det(4A)$.

T

**Problem 2**. *5pts.*

Determine whether the following statements are true or false.

(a) There exist invertible $2\times 2$ matrices $A$ and $B$ such that $\det(A+B) = \det(A)+\det(B)$.

T

(b) The trace of any square matrix is the sum of its diagonal entries.

T

(c) If vector $\vec{v}$ is an eigenvector of both $A$ and $B$, then $\vec{v}$ is an eigenvector of $AB$.

T

(d) If matrix $A$ is positive definite, then all the eigenvalues of $A$ must be positive.

T

(e) The function $q(x_1, x_2) = 3x_1^2 + 4x_1x_2 + 5x_2$ is a quadratic form.

F

**Problem 3**. *7pts.*

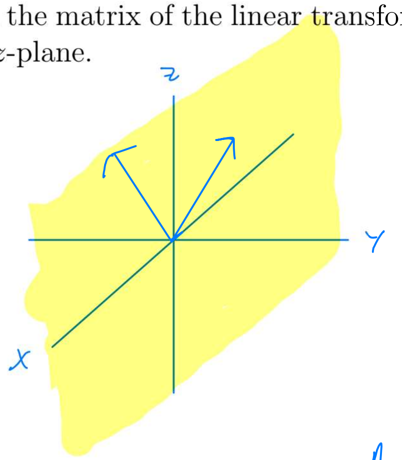
Consider a linear system of four equations with three unknowns. We are told that the system has a unique solution. What does the reduced row-echelon form of the coefficient matrix of this system look like? Explain your answer.

Since the system has a unique solution, the rank must equal the number of unknowns. The reduced row-echelon form of this $4 \times 3$ matrix must then be

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

**Problem 4**. *8pts.*

Find the matrix of the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ given by reflection about the $x-z$-plane.




x and z components stay the same

y component is reversed

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Problem 5**. *8pts.*

Find the inverse of the linear transformation

$$T\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = x_1\begin{bmatrix}22\\-16\\8\\5\end{bmatrix} + x_2\begin{bmatrix}13\\-3\\9\\4\end{bmatrix} + x_3\begin{bmatrix}8\\-2\\7\\3\end{bmatrix} + x_4\begin{bmatrix}3\\-2\\2\\1\end{bmatrix}$$

from $\mathbb{R}^4$ to $\mathbb{R}^4$.

$$\left[\begin{array}{cccc|cccc}22&13&8&3&1&0&0&0\\-16&-3&-2&-2&0&1&0&0\\8&9&7&2&0&0&1&0\\5&4&3&1&0&0&0&1\end{array}\right]\div 22 \quad \left[\begin{array}{cccc|cccc}1&\frac{13}{22}&\frac{4}{11}&\frac{3}{22}&\frac{1}{22}&0&0&0\\-16&-3&-2&-2&0&1&0&0\\8&9&7&2&0&0&1&0\\5&4&3&1&0&0&0&1\end{array}\right]\begin{array}{l}\\+16\,\mathrm{I}\\-8\,\mathrm{I}\\-5\,\mathrm{I}\end{array}$$

$$\left[\begin{array}{cccc|cccc}1&\frac{13}{22}&\frac{4}{11}&\frac{3}{22}&\frac{1}{22}&0&0&0\\0&\frac{71}{11}&\frac{42}{11}&\frac{2}{11}&\frac{8}{11}&1&0&0\\0&\frac{47}{11}&\frac{45}{11}&\frac{10}{11}&-\frac{4}{11}&0&1&0\\0&\frac{23}{22}&\frac{13}{11}&\frac{7}{22}&-\frac{5}{22}&0&0&1\end{array}\right]\times\frac{11}{71} \quad \left[\begin{array}{cccc|cccc}1&\frac{13}{22}&\frac{4}{11}&\frac{3}{22}&\frac{1}{22}&0&0&0\\0&1&\frac{42}{71}&\frac{2}{71}&\frac{8}{71}&\frac{11}{71}&0&0\\0&\frac{47}{11}&\frac{45}{11}&\frac{10}{11}&-\frac{4}{11}&0&1&0\\0&\frac{23}{22}&\frac{13}{11}&\frac{7}{22}&-\frac{5}{22}&0&0&1\end{array}\right]\begin{array}{l}-\frac{13}{22}\,\mathrm{II}\\ \\-\frac{47}{11}\,\mathrm{II}\\-\frac{23}{22}\,\mathrm{II}\end{array}$$

$$\left[\begin{array}{cccc|cccc}1&0&\frac{1}{71}&\frac{17}{142}&-\frac{3}{142}&-\frac{13}{142}&0&0\\0&1&\frac{42}{71}&\frac{2}{71}&\frac{8}{71}&\frac{11}{71}&0&0\\0&0&\frac{111}{71}&\frac{56}{71}&-\frac{60}{71}&-\frac{47}{71}&1&0\\0&0&\frac{40}{71}&\frac{41}{142}&-\frac{49}{142}&-\frac{23}{142}&0&1\end{array}\right]\times\frac{71}{111} \quad \left[\begin{array}{cccc|cccc}1&0&\frac{1}{71}&\frac{17}{142}&-\frac{3}{142}&-\frac{13}{142}&0&0\\0&1&\frac{42}{71}&\frac{2}{71}&\frac{8}{71}&\frac{11}{71}&0&0\\0&0&1&\frac{56}{111}&-\frac{20}{37}&-\frac{47}{111}&\frac{71}{111}&0\\0&0&\frac{40}{71}&\frac{41}{142}&-\frac{49}{142}&-\frac{23}{142}&0&1\end{array}\right]\begin{array}{l}-\frac{1}{71}\,\mathrm{III}\\-\frac{42}{71}\,\mathrm{III}\\ \\-\frac{40}{71}\,\mathrm{III}\end{array}$$

$$\left[\begin{array}{cccc|cccc}1&0&0&\frac{25}{222}&-\frac{1}{74}&-\frac{19}{222}&-\frac{1}{111}&0\\0&1&0&-\frac{10}{37}&\frac{16}{37}&\frac{15}{37}&-\frac{14}{37}&0\\0&0&1&\frac{56}{111}&-\frac{20}{37}&-\frac{47}{111}&\frac{71}{111}&0\\0&0&0&\frac{1}{222}&-\frac{3}{74}&\frac{17}{222}&-\frac{40}{111}&1\end{array}\right]\times 222 \quad \left[\begin{array}{cccc|cccc}1&0&0&\frac{25}{222}&-\frac{1}{74}&-\frac{19}{222}&-\frac{1}{111}&0\\0&1&0&-\frac{10}{37}&\frac{16}{37}&\frac{15}{37}&-\frac{14}{37}&0\\0&0&1&\frac{56}{111}&-\frac{20}{37}&-\frac{47}{111}&\frac{71}{111}&0\\0&0&0&1&-9&17&-80&222\end{array}\right]\begin{array}{l}-\frac{25}{222}\,\mathrm{IV}\\+\frac{10}{37}\,\mathrm{IV}\\-\frac{56}{111}\,\mathrm{IV}\\ \end{array}$$

$$\left[\begin{array}{cccc|cccc}1&0&0&0&1&-2&9&-25\\0&1&0&0&-2&5&-22&60\\0&0&1&0&4&-9&41&-112\\0&0&0&1&-9&17&-80&222\end{array}\right]$$

$$\boxed{T^{-1}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = x_1\begin{bmatrix}1\\-2\\4\\9\end{bmatrix} + x_2\begin{bmatrix}-2\\5\\-9\\17\end{bmatrix} + x_3\begin{bmatrix}9\\-22\\41\\-80\end{bmatrix} + x_4\begin{bmatrix}-25\\60\\-112\\222\end{bmatrix}}$$

**Problem 6**. *7pts.*

Consider the plane $2x_1 - 3x_2 + 4x_3 = 0$. Find a basis $\mathfrak{B}$ of this plane such that

$$\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

$\vec{x} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$, $[x]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow B \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$, $B = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$

$2a + 3d = 2$
$2b + 3e = 0$
$2c + 3f = -1$

$2a - 3b + 4c = 0$
$2d - 3e + 4f = 0$

$$\left[\begin{array}{cccccc|c} 2 & 0 & 0 & 3 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 3 & -1 \\ 2 & -3 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -3 & 4 & 0 \end{array}\right] \begin{array}{l} \div 2 \\ \div 2 \\ \div 2 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & \frac{3}{2} & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{3}{2} & -\frac{1}{2} \\ 2 & -3 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -3 & 4 & 0 \end{array}\right] \begin{array}{l} \\ \\ \\ -2\text{I} \\ \\ \end{array} \left[\begin{array}{cccccc|c} 1 & 0 & 0 & \frac{3}{2} & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & -3 & 4 & -3 & 0 & 0 & -2 \\ 0 & 0 & 0 & 2 & -3 & 4 & 0 \end{array}\right] \begin{array}{l} \\ \\ \\ +3\text{II} \\ \\ \end{array} \left[\begin{array}{cccccc|c} 1 & 0 & 0 & \frac{3}{2} & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 4 & -3 & \frac{9}{2} & 0 & -2 \\ 0 & 0 & 0 & 2 & -3 & 4 & 0 \end{array}\right] \begin{array}{l} \\ \\ \\ -4\text{III} \\ \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & \frac{3}{2} & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & -3 & \frac{9}{2} & -6 & 0 \\ 0 & 0 & 0 & 2 & -3 & 4 & 0 \end{array}\right] \begin{array}{l} \\ \\ \\ \div -3 \\ \\ \end{array} \left[\begin{array}{cccccc|c} 1 & 0 & 0 & \frac{3}{2} & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & -\frac{3}{2} & 2 & 0 \\ 0 & 0 & 0 & 2 & -3 & 4 & 0 \end{array}\right] \begin{array}{l} -\frac{3}{2}\text{IV} \\ \\ \\ \\ -2\text{IV} \end{array} \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & \frac{9}{4} & -3 & 1 \\ 0 & 1 & 0 & 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & -\frac{3}{2} & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$a + \frac{9}{4}e - 3f = 1$
$b + \frac{3}{2}e = 0$
$c + \frac{3}{2}f = -\frac{1}{2}$
$d - \frac{3}{2}e + 2f = 0$

Let $e = 4, f = 2$

$\Rightarrow$
$a + 9 - 6 = 1 \Rightarrow a = -2$
$b + 6 = 0 \Rightarrow b = -6$
$c + 3 = -\frac{1}{2} \Rightarrow c = -\frac{7}{2}$
$d - 6 + 4 = 0 \Rightarrow d = 2$

$$\boxed{B = \left\{ \begin{bmatrix} -2 \\ -6 \\ -7/2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} \right\}}$$

**Problem 7**. *8pts.*

Find the matrix $B$ of the linear transformation

$$T(\vec{x}) = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \vec{x} \quad \text{with respect to the basis} \quad \vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Let $S = [\vec{v}_1 \ \vec{v}_2] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $S^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

then

$$B = S^{-1} A S = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\boxed{B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$$

**Problem 8**. *8pts.*

Find the $QR$ factorization of the matrix

$$\underbrace{\begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}}_{Q}\underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{R} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

$\vec{u}_1 \quad \vec{u}_2 \quad \vec{u}_3 \qquad\qquad \vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3$

$r_{11} = \|\vec{v}_1\| = \sqrt{1+1+1+1} = 2$, $\quad \vec{u}_1 = \frac{1}{r_{11}}\vec{v}_1 = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

$r_{12} = \vec{u}_1\cdot\vec{v}_2 = \begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}\cdot\begin{bmatrix}1\\0\\0\\1\end{bmatrix} = 1$, $\quad \vec{v}_2^{\perp} = \vec{v}_2 - r_{12}u_1 = \begin{bmatrix}1\\0\\0\\1\end{bmatrix} - \begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix} = \begin{bmatrix}1/2\\-1/2\\-1/2\\1/2\end{bmatrix}$

$r_{22} = \|\vec{v}_2^{\perp}\| = \sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} = 1$, $\quad \vec{u}_2 = \frac{1}{r_{22}}\vec{v}_2^{\perp} = \begin{bmatrix}1/2\\-1/2\\-1/2\\1/2\end{bmatrix}$

$r_{13} = \vec{u}_1\cdot\vec{v}_3 = \begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}\cdot\begin{bmatrix}0\\2\\1\\-1\end{bmatrix} = 1$, $\quad r_{23} = \vec{u}_2\cdot\vec{v}_3 = \begin{bmatrix}1/2\\-1/2\\-1/2\\1/2\end{bmatrix}\cdot\begin{bmatrix}0\\2\\1\\-1\end{bmatrix} = -2$, $\quad \vec{v}_3^{\perp} = \vec{v}_3 - r_{13}\vec{u}_1 - r_{23}\vec{u}_2 = \begin{bmatrix}0\\2\\1\\-1\end{bmatrix} - \begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix} - \begin{bmatrix}-1\\1\\1\\-1\end{bmatrix} = \begin{bmatrix}1/2\\1/2\\-1/2\\-1/2\end{bmatrix}$

$r_{13} = \|\vec{v}_3^{\perp}\| = \sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} = 1$, $\quad \vec{u}_3 = \frac{1}{r_{33}}\vec{v}_3^{\perp} = \begin{bmatrix}1/2\\1/2\\-1/2\\-1/2\end{bmatrix}$

**Problem 9**. *8pts.*

Find all the least-squares solutions $\vec{x}^*$ of the system $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}.$$

Draw a sketch showing the vector $\vec{b}$, the image of $A$, the vector $A\vec{x}^*$, and the vector $\vec{b} - A\vec{x}^*$.

$\det A = 0$, Solve

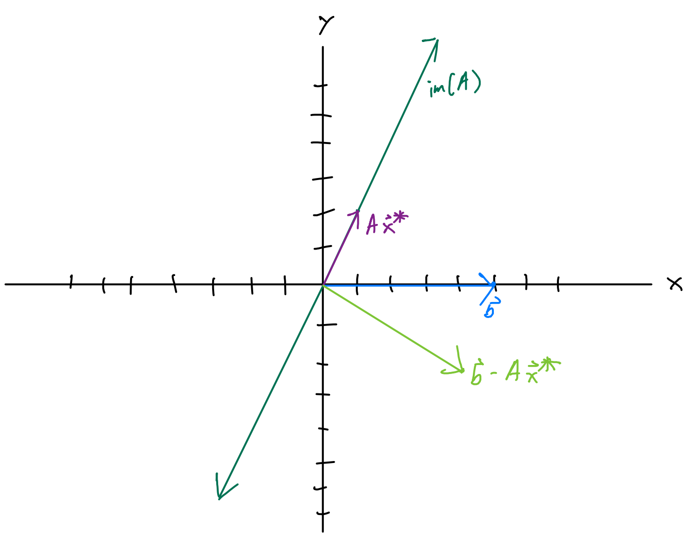
$$A^T A \vec{x} = A^T \vec{b} \Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 5 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5x + 15y \\ 15x + 45y \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \end{bmatrix} \qquad y = \tfrac{1}{3} - \tfrac{1}{3}x$$

$$\left[\begin{array}{cc|c} 5 & 15 & 5 \\ 15 & 45 & 15 \end{array}\right] \xrightarrow{\div 5} \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 15 & 45 & 15 \end{array}\right] \xrightarrow{-15\mathrm{I}} \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 0 & 0 \end{array}\right] \rightarrow x + 3y = 1$$

Let $y = a$, then $\boxed{\vec{x}^* \text{ is of the form } \begin{bmatrix} 1-3a \\ a \end{bmatrix} \text{ for } a \in \mathbb{R}.}$

$$A\vec{x}^* = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 1-3a \\ a \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{b} - A\vec{x}^* = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$



**Problem 10**. *8pts.*

Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \\ 1 & 8 & 27 & 64 & 125 \\ 1 & 16 & 81 & 256 & 625 \end{bmatrix}.$$

| Scalars | Swaps |
|---|---|
| 2 | |
| 6 | |

$$\begin{array}{cc} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \\ 1 & 8 & 27 & 64 & 125 \\ 1 & 16 & 81 & 256 & 625 \end{bmatrix} & \begin{array}{c} \\ -\mathrm{I} \\ -\mathrm{I} \\ -\mathrm{I} \\ -\mathrm{I} \end{array} \end{array} \begin{array}{cc} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 3 & 8 & 15 & 24 \\ 0 & 7 & 26 & 63 & 124 \\ 0 & 15 & 80 & 255 & 624 \end{bmatrix} & \begin{array}{c} \\ \\ -3\mathrm{II} \\ -7\mathrm{II} \\ -15\mathrm{II} \end{array} \end{array}$$

$$\begin{array}{cc} \begin{array}{c} \\ \\ \\ -7\mathrm{II} \\ -15\mathrm{II} \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 6 & 12 \\ 0 & 0 & 12 & 42 & 96 \\ 0 & 0 & 50 & 210 & 564 \end{bmatrix} \end{array} \div 2 \begin{array}{cc} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 12 & 42 & 96 \\ 0 & 0 & 50 & 210 & 564 \end{bmatrix} & \begin{array}{c} \\ \\ \\ -12\mathrm{III} \\ -50\mathrm{III} \end{array} \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 6 & 24 \\ 0 & 0 & 0 & 60 & 264 \end{bmatrix} \div 6 \begin{array}{cc} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 60 & 264 \end{bmatrix} & \begin{array}{c} \\ \\ \\ \\ -60\mathrm{IV} \end{array} \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 24 \end{bmatrix} = B$$

$$\det A = 2 \cdot 6 \cdot \det B = 2 \cdot 6 \cdot 24 = \boxed{288}$$

**Problem 11**. *7pts.*

Use Cramer's rule to solve the system

$$\begin{aligned} 2x + 3y &= 8 \\ 4y + 5z &= 3 \\ 6x + 7z &= -1. \end{aligned}$$

$$A\vec{x} = \vec{b} \rightarrow \begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 5 \\ 6 & 0 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \\ -1 \end{bmatrix}$$

$$\det A = 2\cdot4\cdot7 + 3\cdot5\cdot6 + 0 - 0 - 0 - 0 = 56 + 90$$
$$= 146$$

$$A_{\vec{b},1} = \begin{bmatrix} 8 & 3 & 0 \\ 3 & 4 & 5 \\ -1 & 0 & 7 \end{bmatrix}, \quad \det A_{\vec{b},1} = 8\cdot4\cdot7 + 3\cdot5\cdot-1 + 0 - 0 - 0 - 7\cdot3\cdot3 \quad , \quad x = \frac{\det A_{\vec{b},1}}{\det A} = 1$$
$$= 224 - 15 - 63$$
$$= 146$$

$$A_{\vec{b},2} = \begin{bmatrix} 2 & 8 & 0 \\ 0 & 3 & 5 \\ 6 & -1 & 7 \end{bmatrix}, \quad \det A_{\vec{b},2} = 2\cdot3\cdot7 + 8\cdot5\cdot6 + 0 - 0 - -1\cdot5\cdot2 - 0 \quad , \quad y = \frac{\det A_{\vec{b},2}}{\det A} = 2$$
$$= 42 + 240 + 10$$
$$= 292$$

$$A_{\vec{b},3} = \begin{bmatrix} 2 & 3 & 8 \\ 0 & 4 & 3 \\ 6 & 0 & -1 \end{bmatrix}, \quad \det A_{\vec{b},3} = 2\cdot4\cdot-1 + 3\cdot3\cdot6 + 0 - 6\cdot4\cdot8 - 0 - 0 \quad , \quad z = \frac{\det A_{\vec{b},3}}{\det A} = -1$$
$$= -8 + 54 - 192$$
$$= -146$$

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}$$

**Problem 12**. *7pts.*

Show that 4 is an eigenvalue of

$$A = \begin{bmatrix} -6 & 6 \\ -15 & 13 \end{bmatrix} \qquad A - \lambda I_2 = \begin{bmatrix} -6-\lambda & 6 \\ -15 & 13-\lambda \end{bmatrix}$$

and find all corresponding eigenvectors.

$$\det(A - \lambda I_2) = (-6-\lambda)(13-\lambda) + 90 = -78 - 7\lambda + \lambda^2 + 90$$

$$= \lambda^2 - 7\lambda + 12 = (\lambda-3)(\lambda-4) = 0$$

$$\Rightarrow \lambda = 3, \ \underline{\lambda = 4} \quad \text{so 4 is an eigenvalue}$$

$$A - 4I_2 = \begin{array}{c} \begin{array}{cc} 3 & 5 \end{array} \\ \begin{bmatrix} -10 & 6 \\ -15 & 9 \end{bmatrix} \end{array} \qquad 3\begin{bmatrix} -10 \\ -15 \end{bmatrix} + 5\begin{bmatrix} 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\ker(A - 4I_2) = \boxed{\operatorname{span}\begin{bmatrix} 3 \\ 5 \end{bmatrix}}$$

**Problem 13**. *7pts.*

Suppose a real $3 \times 3$ matrix $A$ has only two distinct eigenvalues. Suppose that $\operatorname{tr}(A) = 1$ and $\det(A) = 3$. Find the eigenvalues of $A$ with their algebraic multiplicities.

$tr(A) = 1 = \lambda_1 + 2\lambda_2$

$det(A) = 3 = \lambda_1 \lambda_2^2$

$1 = 3 + 2(-1) = 1$

$3 = (3)(-1)^2 = 3$

A solution (by guess & check) is

$\lambda_1 = 3, \quad \lambda_2 = -1$

where $almu(3) = 1, \quad almu(-1) = 2$

**Problem 14**. *7pts.*

Find the matrix of the quadratic form $q(x_1, x_2) = 6x_1^2 - 7x_1x_2 + 8x_2^2$.

$$q\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} 6x_1 - \frac{7}{2}x_2 \\ -\frac{7}{2}x_1 + 8x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} 6 & -7/2 \\ -7/2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{x} \cdot A\vec{x}$$

$$\boxed{A = \begin{bmatrix} 6 & -7/2 \\ -7/2 & 8 \end{bmatrix}}$$