Midterm 2
Linear Algebra and Applications
(Math 33A-001)

Answer the questions in the spaces provided. If you run out of room for an answer, please continue on the back of the page. **Show all of your work.**

Name: ______________________ U ID: ______________

Question:	1	2	3	Total
Points:	5	5	10	20
Score:	5	5	7	17

TA: Adam Lott Meeting Day: Thursday

1. $\boxed{\text{5 points}}$ Let A be a 3×2 matrix with column vectors $\vec{v}_1, \vec{v}_2$, i.e.,

$$A = \begin{pmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{pmatrix}$$

$\vec{v}_1, \vec{v}_2 \in \mathbb{R}^3$

$A : \mathbb{R}^2 \to \mathbb{R}^3$

Let $\vec{v}$ be a non-zero vector in $\mathbb{R}^3$. You are told that $\vec{v}, \vec{v}_1, \vec{v}_2$ form a basis of $\mathbb{R}^3$. Then what is the **rank** of the matrix B with column vectors $\vec{v}$, $2\vec{v} + \vec{v}_1$, $2\vec{v} + 3\vec{v}_2$,

i.e., $B = \begin{pmatrix} | & | & | \\ \vec{v} & 2\vec{v} + \vec{v}_1 & 2\vec{v} + 3\vec{v}_2 \\ | & | & | \end{pmatrix}$?

(**Remark:** An answer without proper justification ears you a **'0' point**. You must justify your answer.)

$\vec{v} \neq 0$, a $B_{\mathbb{R}^3} = \{\vec{v}, \vec{v}_1, \vec{v}_2\}$.

$\vec{v}, \vec{v}_1, \vec{v}_2$ are linearly independent $\therefore$ $c\vec{v} + c_1\vec{v}_1 + c_2\vec{v}_2 = 0$ has only solution $c = c_1 = c_2 = 0$

check if columns of B are linearly dependent or indep:

suppose:

$d_1(\vec{v}) + d_2(2\vec{v} + \vec{v}_1) + d_3(2\vec{v} + 3\vec{v}_2) = 0$

$(d_1 + 2d_2 + 2d_3)\vec{v} + (d_2)\vec{v}_1 + (3d_3)\vec{v}_2 = 0$

$\Rightarrow$ $d_1 + 2d_2 + 2d_3 = 0$

$d_2 = 0 \Rightarrow d_2 = 0$

$3d_3 = 0 \Rightarrow d_3 = 0$

$\Rightarrow d_1 = 0$

$d_1 = d_2 = d_3 = 0$ is the only solution $\therefore$ columns of B are linearly independent

$\Rightarrow$ B has full rank

$\boxed{\text{rank}(B) = 3}$

5

good!

5

2. $\boxed{\text{5 points}}$ Let V be a subspace of $\mathbb{R}^n$ of $\dim V = m$ and $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ a basis of V. Show that a vector $\vec{x} \in \mathbb{R}^n$ is orthogonal to V if it is orthogonal to all the vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$.

$V \underset{\text{(subspace)}}{\subset} \mathbb{R}^n$, $\dim(V) = m$, $\mathcal{B}_V = \{\vec{v}_1, \ldots \vec{v}_m\}$

show $\vec{x} \in \mathbb{R}^n \perp V$ if it is orthogonal to all vectors in $\mathcal{B}_V$

all linear combinations of $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_m$

$V = \{\text{all } \vec{x} \in \mathbb{R}^n : \vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots c_m\vec{v}_m \text{ for all } c_1, \ldots c_m \in \mathbb{R}\}$

if $\vec{x} \perp$ all vectors of $\mathcal{B}_V$, then

$\vec{x} \cdot \vec{v}_1 = 0$, $\vec{x} \cdot \vec{v}_2 = 0$, $\ldots$ $\vec{x} \cdot \vec{v}_m = 0$

and $\vec{x} \perp$ all scaled vectors of $\mathcal{B}_V$ $(c_1, c_2, \ldots c_m \in \mathbb{R})$

$\vec{x} \cdot c_1\vec{v}_1 = 0$, $\vec{x} \cdot c_2\vec{v}_2 = 0$, $\ldots$ $\vec{x} \cdot c_m\vec{v}_m = 0$

$\Rightarrow (\vec{x} \cdot c_1\vec{v}_1) + \ldots + (\vec{x} \cdot c_m\vec{v}_m) = 0$

$\vec{x} \cdot (\underbrace{c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots c_m\vec{v}_m}_{\text{any vector } \in V \text{ by definition}}) = 0$

$\therefore$ if $\vec{x} \perp \vec{v}_1, \vec{v}_2, \ldots \vec{v}_m$, $\vec{x} \perp V$

3. $\boxed{\text{10 points}}$ For each of the following statements, determine whether it is true or false.

0 1. Let A and B be two $n \times n$ matrices. You are told that $A + B$ is **invertible**. Then A and B are necessarily invertible.

1 2. Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a set of **linearly independent** vectors of $\mathbb{R}^n$. Let $T = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_l\}$ be another set of **linearly independent** vectors of $\mathbb{R}^n$. Then $S \cup T$ is always a linearly independent set of vectors.

2 3. Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a set of **orthonormal** vectors of $\mathbb{R}^n$. Let $T = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_l\}$ be another set of **orthonormal** vectors of $\mathbb{R}^n$. Then $S \cup T$ is always an orthonormal set of vectors.

2 4. Let A and B be two $n \times n$ matrices such that $\text{rank}(AB) < n$. You are told that A is **invertible**. Then B is never invertible.

2 5. Let A and B be two $n \times n$ matrices such that $\text{rank}(A) = \text{rank}(B)$. Then $\text{Ker}(A) = \text{Ker}(B)$ always holds.

(**Remark:** Only a 'True' or 'False' answer without any justification ears you a **'0' point**. You must justify your answer.)

3.

~~1.~~ $A_{n\times n}$, $B_{n\times n}$, $A+B$ is invertible $\Rightarrow$ A is invertible, B is invertible

$\boxed{\text{TRUE}}$ $\ker(A+B) = \{\vec{0}\}$ $\Rightarrow$ $(A+B)\vec{x} = 0$ has only the

solution $\vec{x} = \vec{0}$. Rearrange:

$(A+B)\vec{x} = \vec{0}$ $\Leftarrow$ $\ker(A+B) = \{\vec{0}\}$

$\ker(A) = \ker(B)$ $\Leftarrow$ $A\vec{x} + B\vec{x} = \vec{0}$ $\Leftarrow$ equivalent equation has only solution $\vec{x} = \vec{0}$

$= \{\vec{0}\}$ $\Rightarrow \begin{cases} A\vec{x} = \vec{0} & \text{has only the solution } \vec{x} = \vec{0} \\ B\vec{x} = \vec{0} & \text{has only the solution } \vec{x} = \vec{0} \end{cases}$

$\ker(A) = \ker(B) = \{\vec{0}\}$ $\therefore$ A & B are invertible

2. $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$, lin indep, $\in \mathbb{R}^n$

$T = \{\vec{w}_1, \ldots, \vec{w}_l\}$, lin indep, $\in \mathbb{R}^n$

Give counterexample -1

$S \cup T = \{\vec{v}_1, \ldots \vec{v}_k, \vec{w}_1 \ldots \vec{w}_l\}$ is always a lin indep?

$\boxed{\text{FALSE}}$ Consider the case that $\vec{w}_1 = \vec{v}_1$

$c_1\vec{v}_1 + \ldots c_k\vec{v}_k + d_1\vec{w}_1 + \ldots d_l\vec{w}_l = 0$ has a non trivial

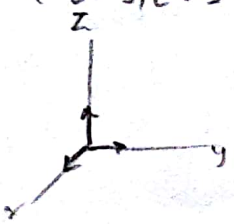
solution $c_1 = 1$, $d_1 = -1$, $c_2, c_3, \ldots, c_k, d_2, d_3, \ldots, d_l = 0$

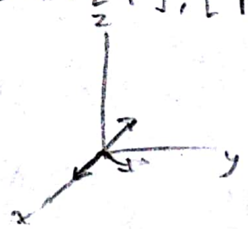
3. S & T are orthonormal sets in $\mathbb{R}^n$

$S \cup T$ is always a orthonormal?

$\boxed{\text{FALSE}}$ Counterexample: consider the case in $\mathbb{R}^3$ that

$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, $T = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$, $S \cup T = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$





$\vec{v}_1 \cdot \vec{v}_2 = 0$ $\|\vec{v}_1\| = 1$

$\vec{v}_2 \cdot \vec{v}_3 = 0$ $\|\vec{v}_2\| = 1$

$\vec{v}_3 \cdot \vec{v}_1 = 0$ $\|\vec{v}_3\| = 1$

$\vec{w}_1 \cdot \vec{w}_2 = 0$ $\|\vec{w}_1\| = 1$

$\vec{w}_2 \cdot \vec{w}_3 = 0$ $\|\vec{w}_2\| = 1$

$\vec{w}_3 \cdot \vec{w}_1 = 0$ $\|\vec{w}_3\| = 1$

$\vec{v}_1 \cdot \vec{w}_1 \neq 0$

$\vec{v}_2 \cdot \vec{w}_2 \neq 0$

$\vec{v}_3 \cdot \vec{w}_3 \neq 0$

...

not all vectors in $S \cup T$ are mutually $\perp$, therefore $S \cup T$ is not an orthonormal set

4. $A_{n\times n}$, $B_{n\times n}$, $AB_{n\times n}$, $\text{rank}(AB) < n$

if A is invertible then B is never invertible.

$\boxed{\text{TRUE}}$ Proof by contradiction: assume B is invertible

$\ker(A) = \{\vec{0}\}$, $\ker(B) = \{\vec{0}\}$

$A\vec{x} = \vec{0}$ for only $\vec{x} = \vec{0}$

$B\vec{x} = \vec{0}$ for only $\vec{x}$

$\ker(AB) = \{$ all $\vec{x}$ that $AB\vec{x} = \vec{0}\}$

suppose $AB\vec{x} = \vec{0} \Rightarrow A(B\vec{x}) = \vec{0}$

$B\vec{x} \in \ker(A) \therefore B\vec{x} = \vec{0}$

$\vec{x} \in \ker B \therefore \vec{x} = \vec{0}$

$\ker(AB) = \{\vec{0}\}$

AB would be invertible $\Rightarrow \text{rank}(AB) = n$ (contradiction)

assumption yields a contradiction (rank(AB) should be $< n$).

$\therefore$ B cannot be invertible

5. $A_{n\times n}$ $B_{n\times n}$, $\text{rank}(A) = \text{rank}(B)$, $\ker(A) = \ker(B)$ always

$\boxed{\text{FALSE}}$ only $\dim(\ker(A)) = \dim(\ker(B))$ holds

counterexample in $\mathbb{R}^2$:

let $A = \text{proj}_{(y=x)}(\vec{x}) = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$, $\text{rank}(A) = 1$

$\ker(A) = \{\begin{bmatrix} x \\ y \end{bmatrix}$ that $A\begin{bmatrix} x \\ y \end{bmatrix} = \vec{0}\}$ $\left[\begin{array}{cc|c} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \end{array}\right] \Rightarrow x + y = 0$

$y = -x$

$\ker(A) = \text{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -x \end{bmatrix} = x\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

let $B = \text{proj}_{(y=0)}\vec{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\text{rank}(B) = \text{rank}(A) = 1$

$\ker(B) = \{\begin{bmatrix} x \\ y \end{bmatrix}$ that $B\begin{bmatrix} x \\ y \end{bmatrix} = \vec{0}\}$ $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x = 0$

$\ker(B) = \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix} = y\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\text{rank}(A) = \text{rank}(B)$, $\ker(A) \neq \ker(B)$