Midterm 1
Linear Algebra and Applications
(Math 33A-001)

Answer the questions in the spaces provided. If you run out of room for an answer, continue on the back of the page. **Show your work.**

Name ______________________ U ID ______________________

TA's Name: Sam Yih TA Meeting Day: Tuesday

Question:	1	2	3	4	Total
Points:	5	5	5	5	20
Score:	5	5	4	2	16

1. $\boxed{\text{5 points}}$ Let T be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that

$$T\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\-1\\0\end{pmatrix},\quad T\begin{pmatrix}-1\\1\\-1\end{pmatrix}=\begin{pmatrix}3\\2\\-6\end{pmatrix}\quad\text{and}\quad T\begin{pmatrix}-1\\-1\\2\end{pmatrix}=\begin{pmatrix}-1\\0\\4\end{pmatrix}.$$

Compute the matrix of T.

The matrix of T $A = \begin{bmatrix} | & | & | \\ T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \\ | & | & | \end{bmatrix}$, so we know that $A = \begin{bmatrix} 1 & a & d \\ -1 & b & e \\ 0 & c & f \end{bmatrix}$

We also know that $\begin{bmatrix} 1 & a & d \\ -1 & b & e \\ 0 & c & f \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -6 \end{bmatrix} \to$

$-1+a-d=3 \to d=a-4$
$1+b-e=2 \to e=b-1$
$0+c-f=-6 \to f=c+6$

and $\begin{bmatrix} 1 & a & d \\ -1 & b & e \\ 0 & c & f \end{bmatrix}\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} \to$

$-1-a+2d=-1$
$1-b+2e=0$
$0-c+2f=4$

We solve for the unknowns:

$-1-a+2(a-4)=-1$
$-a+2a-8=0$
$a=8$
$d=8-4=4$

$1-b+2(b-1)=0$
$-b+2b-2=-1$
$b=1$
$e=1-1=0$

$-c+2(c+6)=4$
$-c+2c+12=4$
$c=-8$
$f=-8+6=-2$

So the matrix of T $= \boxed{\begin{bmatrix} 1 & 8 & 4 \\ -1 & 1 & 0 \\ 0 & -8 & -2 \end{bmatrix}}$

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2. 5 points Find a 3×3 matrix A such that $A\vec{x}$ is parallel to the vector $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ for all $\vec{x} \in \mathbb{R}^3$.

We find the unit vector $\vec{u}$ in the direction of $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$: $\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

We then find $\text{proj}_L(\vec{x}) = (\vec{x} \cdot \vec{u})\vec{u}$ (where L is the line containing $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$)

$$= \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}\right)\begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} = \left(\frac{x_1}{\sqrt{3}} + -\frac{x_2}{\sqrt{3}} + \frac{x_3}{\sqrt{3}}\right)\begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{x_1}{3} - \frac{x_2}{3} + \frac{x_3}{3} \\ \frac{-x_1}{3} + \frac{x_2}{3} - \frac{x_3}{3} \\ \frac{x_1}{3} - \frac{x_2}{3} + \frac{x_3}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{1}{3} & \frac{-1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

So $A = \boxed{\begin{bmatrix} \frac{1}{3} & \frac{-1}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{1}{3} & \frac{-1}{3} \\ \frac{1}{3} & \frac{-1}{3} & \frac{1}{3} \end{bmatrix}}$

3. $\boxed{\text{5 points}}$ Let A be a 4×4 matrix, $\vec{b}$ is a non-zero vector in $\mathbb{R}^4$, and $\vec{0}$ is the zero vector in $\mathbb{R}^4$. We are told that the linear system $A\vec{x} = \vec{0}$ has ***infinitely*** many solutions. What can you say about the number of solutions of the system $A\vec{x} = \vec{b}$? You must explain your answer.

We can say that $A\vec{x} = \vec{b}$ has either infinitely many solutions or no solution
The fact that $A\vec{x} = \vec{0}$ has infinitely many solutions means that A is not
4 invertible, so rank$(A) < 4$. This means that $A\vec{x} = \vec{b}$ can have either
infinitely many solutions or no solution.

5 points Let A be a $m \times n$ matrix and B a $n \times p$ matrix. If $\ker(A) = \{\vec{0}\}$ and $\ker(B) = \{\vec{0}\}$, then prove that $\ker AB = \{\vec{0}\}$.

not square

Since $\ker(A) = \{\vec{0}\}$ and $\ker(B) = \{\vec{0}\}$, A and B ~~are invertible~~ matrices. This means that their product AB ~~is also~~ invertible. This in turn means that $\ker(AB) = \{\vec{0}\}$, because if AB is an invertible matrix, then $AB\vec{x} = \vec{0}$ has only one solution, $\vec{x} = \vec{0}$.