20W-MATH33A-1 Final Exam

NEIL VAISHAMPAYAN

TOTAL POINTS

97 / 100

QUESTION 1

1 Problem 1 10 / 10

✓ **- 0 pts** Correct

- 8 pts Some work

- 3 pts One Correct Vector

QUESTION 2

2 Probelm 2 10 / 10

✓ **- 0 pts** Correct

- 5 pts Half Right

- 1 pts Sign Error

- 10 pts Incorrect

- 8 pts What is the basis?

QUESTION 3

3 Problem 3 10 / 10

✓ **- 0 pts** Correct

- 1 pts 1 numerical error

- 2 pts 1 gram-schmidt error

- 10 pts no progress/incorrect method

- 8 pts gram schmidt in wrong order

- 2 pts final answer is illegible but working is correct

- 4 pts major numerical and gram schmidt errors

- 3 pts method to compute R incorrect

- 3 pts Q not an orthogonal matrix

QUESTION 4

4 Problem 4 10 / 10

✓ **- 0 pts** Correct

- 5 pts half correct method

- 8 pts incorrect method

- 1 pts wrong sign

- 2 pts forgot to normalise

- 1 pts numerical error

- 10 pts no submission

QUESTION 5

5 Problem 5 10 / 10

✓ **- 0 pts** Correct

- 1 pts Computational error

- 7 pts Used dot product formula for coordinates in non-orthogonal basis

- 7 pts Used A instead of Q.

- 0 pts Incomplete (variable adjustment)

- 2 pts Several computational errors

- 2 pts Sanity check failure (variable adjustment)

- 10 pts No credit.

QUESTION 6

6 Problem 6 10 / 10

✓ **- 0 pts** Correct

- 1 pts Computational error

- 1 pts Sanity check failure (variable adjustment)

- 1 pts Unclear (variable adjustment)

- 10 pts No credit

- 5 pts Fit the wrong family of curves

QUESTION 7

7 Problem 7 18 / 20

- 0 pts everything is correct

- 20 pts everything is incorrect

✓ **- 2 pts You did not find an explicit form of A^n.**

- 1 pts The formula for A^n is incorrect.

✓ **- 0 pts the formula for q(p_1,p_2) is incorrect.**

- 1 pts eigenvectors correspond to wrong eigenvalues

- 2 pts You do not use eigenvalues of A to find the Min and Max of q(x,y).

- 2 pts You did not explain why switching to the principal axes coordinates preserves the unit circle.

- 4 pts The explanation you provide for finding Max and Min of q(x,y) is incorrect.

- **2 pts** You did not find the correct Max and Min of q(x,y).
- **2 pts** You do not see the difference between a Max (Min) and the points where it is achieved.
- **4 pts** Did not demonstrate understanding of quadratic forms.
- **1 pts** the first eigenvector is incorrect

QUESTION 8

8 Problem 8 **10 / 10**

✓ **- 0 pts** **Correct**
- **10 pts** Did not find the determinant.
- **3 pts** The Sarrus rule is not applicable.
- **4 pts** Computation incomplete, conclusion incorrect.

QUESTION 9

9 Problem 9 **9 / 10**

- **0 pts** Correct
- **1 pts** The squared length of the second vector is 14, not 15

✓ **- 1 pts** **150.8 rounds to 151**
- **3 pts** Wrong answer.
- **10 pts** Problem not attempted.
- **3 pts** Did not find the answer in the required form.

Problem 1 **10 pts**

Find a basis of the kernel of

$$A = \begin{bmatrix} 2 & -1 & 1 & 0 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 5 & -3 \\ 0 & -1 & -3 & 2 \end{bmatrix}$$

Converting to rref:

$$\begin{bmatrix} 2 & -1 & 1 & 0 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 5 & -3 \\ 0 & -1 & -3 & 2 \end{bmatrix} \text{swap} = \begin{bmatrix} 1 & 1 & 5 & -3 \\ -1 & 1 & 1 & -1 \\ 2 & -1 & 1 & 0 \\ 0 & -1 & -3 & 2 \end{bmatrix} \begin{matrix} \\ +\text{I} \\ -2\text{I} \\ \\ \end{matrix} = \begin{bmatrix} 1 & 1 & 5 & -3 \\ 0 & 2 & 6 & -4 \\ 0 & -3 & -9 & 6 \\ 0 & -1 & -3 & 2 \end{bmatrix} \text{swap} = \begin{bmatrix} 1 & 1 & 5 & -3 \\ 0 & -1 & -3 & 2 \\ 0 & -3 & -9 & 6 \\ 0 & 2 & 6 & -4 \end{bmatrix} \begin{matrix} \\ /-1 \\ /-3 \\ /2 \end{matrix} = \begin{bmatrix} 1 & 1 & 5 & -3 \\ 0 & 1 & 3 & -2 \\ 0 & 1 & 3 & -2 \\ 0 & 1 & 3 & -2 \end{bmatrix} \begin{matrix} \\ \\ -\text{II} \\ -\text{II} \end{matrix}$$

$$= \begin{bmatrix} 1 & 1 & 5 & -3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} -\text{II} \\ \\ \\ \\ \end{matrix} = \begin{array}{c} \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} v_1 & v_2 & t_1 & t_2 \end{matrix} \end{array} \leftarrow \text{rref}$$

Using the variables, the system of equations we get are:

$$\begin{cases} v_1 = -2t_1 + t_2 \\ v_2 = -3t_1 + 2t_2 \\ t_1 = t_1 + 0t_2 \\ t_2 = 0t_1 + t_2 \end{cases} \quad \text{or} \quad \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \end{bmatrix} t_1 + \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} t_2$$

Thus the basis is $\left(\begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right)$

Answer: $\boxed{\begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}}$

1 Problem 1 **10 / 10**

- ✓ **- 0 pts** Correct
- **- 8 pts** Some work
- **- 3 pts** One Correct Vector

Problem 2 **10 pts**

For the matrix A from problem 1, find a basis of the orthogonal complement of the image of A.

$(\mathrm{Im}\, A)^{\perp} = \ker(A^t)$

$$A^t = \begin{bmatrix} 2 & -1 & 1 & 0 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 5 & -3 \\ 0 & -1 & 3 & 2 \end{bmatrix} = A$$

Thus, $\ker(A^t) = \ker(A)$

Thus, $(\mathrm{Im}\, A)^{\perp} = \ker(A)$

$$(\mathrm{Im}\, A)^{\perp} = \mathrm{span}\left(\begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right)$$

Thus, the basis is $\begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

Answer: $\boxed{\begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}}$

2 Probelm 2 **10 / 10**

- ✓ **- 0 pts** **Correct**
- **- 5 pts** Half Right
- **- 1 pts** Sign Error
- **- 10 pts** Incorrect
- **- 8 pts** What is the basis?

Problem 3 **10 pts**

Find the QR decomposition of the matrix

$$M = \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \\ \shortparallel & \shortparallel & \shortparallel \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \end{array} = QR = \begin{bmatrix} | & | & | \\ w_1 & w_2 & w_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{bmatrix}$$

$$V_1 = x_{11} w_1 \rightarrow x_{11} = \|v_1\| = \underline{\sqrt{3}}, \quad w_1 = \frac{v_1}{x_{11}} = \underline{\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}}$$

$$V_2 = x_{12} w_1 + x_{22} w_2 \rightarrow x_{12} = V_2 \cdot w_1 = \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \underline{\frac{2}{\sqrt{3}}} \quad x_{22} = \|v_2 - x_{12} w_1\| = \left\| \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} \right\| = \left\| \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \right\| = \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{1}{9}} = \underline{\frac{\sqrt{6}}{3}}$$

$$W_2 = \frac{\begin{bmatrix} -\frac{2}{3} \\ 1/3 \\ 1/3 \end{bmatrix}}{\frac{\sqrt{6}}{3}} = \underline{\begin{bmatrix} -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}}$$

$$V_3 = x_{13} w_1 + x_{23} W_2 + x_{33} W_3 \rightarrow x_{13} = V_3 \cdot w_1 = \underline{-\frac{1}{\sqrt{3}}}, \; x_{23} = V_3 \cdot w_2 = \underline{\frac{-1}{\sqrt{6}}}$$

$$x_{33} = \|v_3 - x_{12} w_1 - x_{23} w_2\| = \left\| \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} -1/3 \\ -1/3 \\ -1/3 \end{bmatrix} - \begin{bmatrix} 1/3 \\ -1/6 \\ -1/6 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \right\| = \sqrt{\frac{1}{4} + \frac{1}{4}} = \underline{\frac{\sqrt{2}}{2}}$$

$$W_3 = \frac{\begin{bmatrix} 0 \\ 1/2 \\ -1/2 \end{bmatrix}}{\frac{\sqrt{2}}{2}} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$\text{Thus } QR = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & \frac{2}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{\sqrt{6}}{3} & \frac{-1}{\sqrt{6}} \\ 0 & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$$

Answer: ______________________________

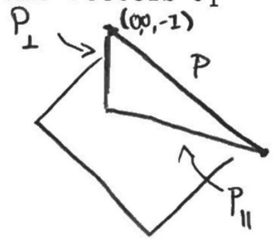
3 Problem 3 **10 / 10**

- ✓ **- 0 pts** **Correct**
- **- 1 pts** 1 numerical error
- **- 2 pts** 1 gram-schmidt error
- **- 10 pts** no progress/incorrect method
- **- 8 pts** gram schmidt in wrong order
- **- 2 pts** final answer is illegible but working is correct
- **- 4 pts** major numerical and gram schmidt errors
- **- 3 pts** method to compute R incorrect
- **- 3 pts** Q not an orthogonal matrix

Problem 4 **10 pts**

Find the distance from the point $P = (0, 0, -1)$ to the plane spent by the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Sketch:



Thus, the shortest distance is $P - P_\parallel$ or $P - \text{Proj}_\pi P$

From the previous problem we know that QR for $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{3} & \frac{2}{\sqrt{3}} \\ 0 & \sqrt{6}/3 \end{bmatrix}$

$$\text{Proj}_\pi = QQ^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac{1}{3}+\frac{2}{3} & \frac{1}{3}-\frac{1}{3} & \frac{1}{3}-\frac{1}{3} \\ \frac{1}{3}-\frac{1}{3} & \frac{1}{3}+\frac{1}{6} & \frac{1}{3}+\frac{1}{6} \\ \frac{1}{3}-\frac{1}{3} & \frac{1}{3}+\frac{1}{6} & \frac{1}{3}+\frac{1}{6} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Thus, $\text{Proj}_\pi P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$, $P - P_\parallel = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$

$$\text{distance} = \left\| \begin{bmatrix} 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \right\| = \sqrt{\frac{1}{4}+\frac{1}{4}} = \frac{\sqrt{2}}{2}$$

Answer: $\boxed{\frac{\sqrt{2}}{2}}$

4 Problem 4 10 / 10

✓ **- 0 pts** Correct

- 5 pts half correct method

- 8 pts incorrect method

- 1 pts wrong sign

- 2 pts forgot to normalise

- 1 pts numerical error

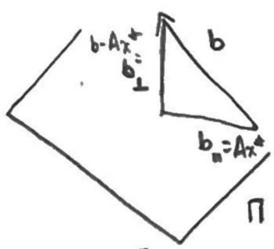
- 10 pts no submission

Problem 5 **10 pts**

Find the orthogonal projection of the vector $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \end{bmatrix}$ on the plane π

spanned by the vectors $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 9 \\ 25 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 5 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

Sketch:



This sketch can be solved as least-squares problem, as we can represent $proj_\pi b = b_\pi = Ax^*$

$$A^tA = \begin{bmatrix} 0 & 1 & 9 & 25 \\ 0 & 1 & 3 & 5 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 9 & 3 & 1 \\ 25 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 707 & 153 & 35 \\ 153 & 35 & 9 \\ 35 & 9 & 4 \end{bmatrix}, \quad A^t\vec{b} = \begin{bmatrix} 0 & 1 & 9 & 25 \\ 0 & 1 & 3 & 5 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 118 \\ 26 \\ 7 \end{bmatrix}$$

Solve $[A^tA \mid A^t\vec{b}]$

$$\left[\begin{array}{ccc|c} 707 & 153 & 35 & 118 \\ 153 & 35 & 9 & 26 \\ 35 & 9 & 4 & 7 \end{array}\right] \xrightarrow{\text{swap}} \left[\begin{array}{ccc|c} 35 & 9 & 4 & 7 \\ 153 & 35 & 9 & 26 \\ 707 & 153 & 35 & 118 \end{array}\right] /35 = \left[\begin{array}{ccc|c} 1 & \frac{9}{35} & \frac{4}{35} & \frac{1}{5} \\ 153 & 35 & 9 & 26 \\ 707 & 153 & 35 & 118 \end{array}\right] \begin{array}{l} \\ -153\,\mathrm{I} \\ -707\,\mathrm{I} \end{array} = \left[\begin{array}{ccc|c} 1 & 9/35 & 4/35 & \frac{1}{5} \\ 0 & -152/35 & -297/35 & -23/5 \\ 0 & -1008/35 & -1603/35 & -117/5 \end{array}\right] \cdot -\frac{35}{152}$$

$$= \left[\begin{array}{ccc|c} 1 & 9/35 & 4/35 & 1/5 \\ 0 & 1 & 297/152 & 161/152 \\ 0 & -\frac{1008}{35} & -\frac{1603}{35} & -117/5 \end{array}\right] + \frac{1008}{35}\,\mathrm{II} = \left[\begin{array}{ccc|c} 1 & 9/35 & 4/35 & 1/5 \\ 0 & 1 & \frac{297}{152} & \frac{161}{152} \\ 0 & 0 & \frac{199}{19} & \frac{135}{19} \end{array}\right] \cdot \frac{19}{199} = \left[\begin{array}{ccc|c} 1 & 9/35 & 4/35 & \frac{1}{5} \\ 0 & 1 & \frac{297}{152} & \frac{161}{152} \\ 0 & 0 & 1 & \frac{135}{199} \end{array}\right] \begin{array}{l} -\frac{4}{35}\,\mathrm{III} \\ -\frac{297}{152}\,\mathrm{III} \\ \\ \end{array}$$

Continued on Back of Page

Answer: $\boxed{Proj_\pi \vec{b} = \begin{bmatrix} 135/199 \\ 120/199 \\ 318/199 \\ 820/199 \end{bmatrix}}$

$$\left[\begin{array}{ccc|c} 1 & 9/35 & 0 & \frac{853}{6965} \\ 0 & 1 & 0 & -\frac{53}{199} \\ 0 & 0 & 1 & \frac{135}{199} \end{array}\right] -\frac{9}{35}\text{II} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{38}{199} \\ 0 & 1 & 0 & -\frac{53}{199} \\ 0 & 0 & 1 & \frac{135}{199} \end{array}\right]$$

Thus $x^* = \underline{\begin{bmatrix} \frac{38}{199} \\ -\frac{53}{199} \\ \frac{135}{199} \end{bmatrix}}$

$$\text{Proj}_\Pi b = b_{\parallel} = Ax^*$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 9 & 3 & 1 \\ 25 & 5 & 1 \end{bmatrix} \begin{bmatrix} \frac{38}{199} \\ -\frac{53}{199} \\ \frac{135}{199} \end{bmatrix} = \begin{bmatrix} \frac{135}{199} \\ \frac{38}{199} - \frac{53}{199} + \frac{135}{199} \\ \frac{9\cdot 38}{199} - \frac{3\cdot 53}{199} + \frac{135}{199} \\ \frac{25\cdot 38}{199} - \frac{5\cdot 53}{199} + \frac{135}{199} \end{bmatrix} = \boxed{\begin{bmatrix} \frac{135}{199} \\ \frac{120}{199} \\ \frac{318}{199} \\ \frac{820}{199} \end{bmatrix}} \quad \square$$

5 Problem 5 10 / 10

- ✓ **- 0 pts** **Correct**
- **- 1 pts** Computational error
- **- 7 pts** Used dot product formula for coordinates in non-orthogonal basis
- **- 7 pts** Used A instead of Q.
- **- 0 pts** Incomplete (variable adjustment)
- **- 2 pts** Several computational errors
- **- 2 pts** Sanity check failure (variable adjustment)
- **- 10 pts** No credit.

Problem 6 **10 pts**

Find a quadratic parabola $y = ax^2 + bx + c$ that best fits the points $P_1 = (0,1)$, $P_2 = (1,0)$, $P_3 = (3,2)$, and $P_4 = (5,4)$.

For form $y=ax^2+bx+c$, we can represent as $A\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ y \end{bmatrix}$

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 9 & 3 & 1 \\ 25 & 5 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \end{bmatrix}$$

We solved this least-squares equation in problem 5, finding $x^* = \begin{bmatrix} \frac{38}{199} \\ \frac{-53}{199} \\ \frac{135}{199} \end{bmatrix}$

Thus, $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 38/199 \\ -53/199 \\ 135/199 \end{bmatrix}$

So the equation is:

Answer: $\boxed{y = \frac{38}{199}x^2 - \frac{53}{199}x + \frac{135}{199}}$

6 Problem 6 10 / 10

- ✓ **- 0 pts** **Correct**
- **- 1 pts** Computational error
- **- 1 pts** Sanity check failure (variable adjustment)
- **- 1 pts** Unclear (variable adjustment)
- **- 10 pts** No credit
- **- 5 pts** Fit the wrong family of curves

Problem 7

$$A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$$

- Find eigenvalues of A. $\text{char}(A) \to \det(A - \lambda Id) = 0$ **4 pts**

$\begin{vmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = (2-\lambda)(-1-\lambda) - 4 = \lambda^2 - \lambda - 2 - 4 = \lambda^2 - \lambda - 6$

$\lambda^2 - \lambda - 6 = 0 \to (\lambda - 3)(\lambda + 2) = 0$

Answer: $\lambda_1 = \boxed{-2}$, $\lambda_2 = \boxed{3}$

- Find the corresponding eigenvectors of A. **5 pts**

Computing $E_{-2} \to \ker(A + 2\,Id)$

$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \to 2x + y = 0$

Thus $v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Computing $E_3 \to \ker(A - 3\,Id)$

$\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \to x - 2y = 0$

Thus $v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Answer: $\boxed{\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}}$, $\boxed{\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$

The problem continues to the next page.

- Find an explicit formula for A^n. **5 pts**

$$A^n = \underbrace{A \cdot A \cdots A}_{n \text{ times}} = \underbrace{SD\cancel{S^{-1}}\cancel{S}D\cancel{S^{-1}} \cdots \cancel{S}DS^{-1}}_{n \text{ times}} = SD^nS^{-1} = S\,\mathrm{diag}(\lambda_1^n, \lambda_2^n)S^{-1}$$

$$S = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \text{ Compute } S^{-1} \to \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{array}\right]_{+2\mathrm{I}} = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 5 & 2 & 1 \end{array}\right]_{/5} = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{2}{5} & \frac{1}{5} \end{array}\right]^{-2\mathrm{II}} = \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{5} & -\frac{2}{5} \\ 0 & 1 & \frac{2}{5} & \frac{1}{5} \end{array}\right]$$

Thus $S^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

Thus

$$A^n = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} (-2)^n & 0 \\ 0 & 3^n \end{bmatrix} \frac{1}{5} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} (-2)^n & 2(3^n) \\ (-2)^{n+1} & 3^n \end{bmatrix} \frac{1}{5} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} (-2)^n + 4(3^n) & (-2)^{n+1} + 2(3^n) \\ (-2)^{n+1} + 2(3^n) & (-2)^{n+2} + 3^n \end{bmatrix}$$

Answer: $A^n = \begin{bmatrix} \frac{(-2)^n + 4(3^n)}{5} & \frac{(-2)^{n+1} + 2(3^n)}{5} \\ \frac{(-2)^{n+1} + 2(3^n)}{5} & \frac{(-2)^{n+2} + 3^n}{5} \end{bmatrix}$

The problem continues to the next page.

Let $q(x,y) = [x,y]\,A \begin{bmatrix} x \\ y \end{bmatrix}$ and let $S = \{(x,y) : x^2+y^2 \le 1\}$ be the unit circle in $\mathbb{R}^2$ centered at the origin. Find

- the global maximum of $q(x,y)$ over S **3 pts**

$$S = \begin{array}{cc} v_1 & v_2 \\ \shortparallel & \shortparallel \end{array} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

$v_1 \cdot v_2 = 0$, so to create an orthonormal eigen basis, we need to normalize the vectors: $U = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$

$U^{-1} = U^t = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$. Thus $q(x,y) = [x,y] \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$$q(x,y) = \Big[\underbrace{\tfrac{x-2y}{\sqrt{5}}}_{p}, \underbrace{\tfrac{2x+y}{\sqrt{5}}}_{q}\Big] \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{x-2y}{\sqrt{5}} \\ \frac{2x+y}{\sqrt{5}} \end{bmatrix} = [p,q] \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = [-2p, 3q] \begin{bmatrix} p \\ q \end{bmatrix} = -2p^2 + 3q^2$$

Max of $-2p^2+3q^2 = 3$ when $q = \pm 1$, $p = 0$

Answer: Max $q(x,y)|_S =$ $\boxed{3}$

- the global minimum of $q(x,y)$ over S **3 pts**

Using the quadratic form found in the last part,

the min of $-2p^2+3q^2 = -2$ when $p = \pm 1, q = 0$

Answer: Min $q(x,y)|_S =$ $\boxed{-2}$

7 Problem 7 **18 / 20**

- **- 0 pts** everything is correct
- **- 20 pts** everything is incorrect
- ✓ **- 2 pts** **You did not find an explicit form of A^n.**
- **- 1 pts** The formula for A^n is incorrect.
- ✓ **- 0 pts** **the formula for q(p_1,p_2) is incorrect.**
- **- 1 pts** eigenvectors correspond to wrong eigenvalues
- **- 2 pts** You do not use eigenvalues of A to find the Min and Max of q(x,y).
- **- 2 pts** You did not explain why switching to the principal axes coordinates preserves the unit circle.
- **- 4 pts** The explanation you provide for finding Max and Min of q(x,y) is incorrect.
- **- 2 pts** You did not find the correct Max and Min of q(x,y).
- **- 2 pts** You do not see the difference between a Max (Min) and the points where it is achieved.
- **- 4 pts** Did not demonstrate understanding of quadratic forms.
- **- 1 pts** the first eigenvector is incorrect

Problem 8 **10 pts**

Let $A = \begin{bmatrix} 2 & -1 & 1 & 0 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 5 & -3 \\ 0 & -1 & -3 & 2 \end{bmatrix}$ be the matrix from problem 1. Find $\det(A)$.

Start with Gauss-Jordan row reduction

$$\begin{vmatrix} 2 & -1 & 1 & 0 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 5 & -3 \\ 0 & -1 & -3 & 2 \end{vmatrix} \text{Swap} = - \begin{vmatrix} 1 & 1 & 5 & -3 \\ -1 & 1 & 1 & -1 \\ 2 & -1 & 1 & 0 \\ 0 & -1 & -3 & 2 \end{vmatrix} \begin{matrix} \\ +\mathrm{I} \\ -2\mathrm{I} \\ \\ \end{matrix} = - \begin{vmatrix} 1 & 1 & 5 & -3 \\ 0 & 2 & 6 & -4 \\ 0 & -3 & -9 & 6 \\ 0 & -1 & -3 & 2 \end{vmatrix} \begin{matrix} \\ /2 \\ \\ \\ \end{matrix} = -2 \begin{vmatrix} 1 & 1 & 5 & -3 \\ 0 & 1 & 3 & -2 \\ 0 & -3 & -9 & 6 \\ 0 & -1 & -3 & 2 \end{vmatrix} \begin{matrix} -\mathrm{II} \\ \\ +3\mathrm{II} \\ +\mathrm{II} \end{matrix}$$

$$= -2 \begin{vmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

From here, use cofactor expansion

$$\to -2 \left(1 \begin{vmatrix} 1 & 3 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} + 2 \begin{vmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} + 1 \begin{vmatrix} 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \right)$$

$$= -2 (0 + 0 + 0) = \underline{0}$$

We can confirm this result conceptually. Recall that $\det(A) = \sum_{s \in S_n} (-1)^{d(s)} a_{1,s(1)} \cdot a_{2,s(2)} \cdots a_{n,s(n)}$. In an upper triangular matrix, the only pattern that does not include one of the zeroes in the lower triangle is the major diagonal. Thus, a determinant of an upper triangular matrix is $\prod_{i=1}^{n} a_{ii}$. However, in our case, that would result in $\det(A) = -2(1 \cdot 1 \cdot 0 \cdot 0) = 0$

Answer: $\det(A) = \boxed{0}$

8 Problem 8 10 / 10

✓ **- 0 pts** **Correct**

- 10 pts Did not find the determinant.

- 3 pts The Sarrus rule is not applicable.

- 4 pts Computation incomplete, conclusion incorrect.

Problem 9 **10 pts**

Find the angle between the vectors

$$\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v_2} = \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \end{bmatrix}$$

in degrees. Round to the nearest degree.

$$\cos\theta = \frac{V_1 \cdot V_2}{\|V_1\| \|V_2\|} = \frac{-2-6}{\sqrt{6}\sqrt{4+9+1}} = \frac{-8}{\sqrt{6}\sqrt{14}}$$

$$\theta = \arccos\left(\frac{-8}{\sqrt{6}\sqrt{14}}\right) \approx 150.8$$

Answer: the angle between the vectors $\vec{v}_1$ and $\vec{v}_2 \approx$ $\boxed{150^\circ}$

9 Problem 9 **9 / 10**

- **- 0 pts** Correct
- **- 1 pts** The squared length of the second vector is 14, not 15
- ✓ **- 1 pts 150.8 rounds to 151**
- **- 3 pts** Wrong answer.
- **- 10 pts** Problem not attempted.
- **- 3 pts** Did not find the answer in the required form.