Midterm II (B)

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May 14 2018

This test totals 40 points and you get 45 minutes to do it. Answer the questions in the spaces provided on the question sheets. Show work unless the question says otherwise. Good luck!

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ID number :

Discussion section : 4B

Question	Points	Score
1	10	10
2	10	10
3	10	10
4	10	10
Total:	40	40

1. Let $E = \{(1,0),(0,1)\}$ be the usual basis of $\mathbb{R}^2$ and let $B = \{(2,1),(1,2)\}$ be another basis of $\mathbb{R}^2$. Answer the following questions. Show work to get full credit.

(a) (2 points) Find the B-coordinates of the vector which has E-coordinates $(8,1)$.

$$\begin{pmatrix}8\\1\end{pmatrix} = \begin{pmatrix}2&1\\1&2\end{pmatrix}\vec{v} \qquad \left(\begin{array}{cc|cc}2&1&1&0\\1&2&0&1\end{array}\right)$$

$$\begin{pmatrix}2/3&-1/3\\-1/3&2/3\end{pmatrix}\begin{pmatrix}8\\1\end{pmatrix} = \vec{v} \qquad \to \left(\begin{array}{cc|cc}1&-1&1&-1\\1&2&0&1\end{array}\right)$$

$$= \begin{pmatrix}16/3-1/3\\-8/3+2/3\end{pmatrix} \qquad \to \left(\begin{array}{cc|cc}1&-1&1&-1\\0&-3&1&-2\end{array}\right)$$

$$= \boxed{\begin{pmatrix}5\\-2\end{pmatrix}} \qquad \to \left(\begin{array}{cc|cc}1&-1&1&-1\\0&1&-1/3&2/3\end{array}\right)$$

$$\to \left(\begin{array}{cc|cc}1&0&2/3&-1/3\\0&1&-1/3&2/3\end{array}\right)$$

$10-2 = 8$

$5-4=1$

(b) (2 points) Find the E-coordinates of the vector which has B-coordinates $[8,1]$.

$$\begin{pmatrix}2&1\\1&2\end{pmatrix}\begin{pmatrix}8\\1\end{pmatrix} = \begin{pmatrix}16+1\\8+2\end{pmatrix} = \boxed{\begin{pmatrix}17\\10\end{pmatrix}}$$

(c) (6 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map which has matrix $A = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$ with respect to E-coordinates. Find the matrix of T with respect to B-coordinates.

$AS = SB$

$$B = S^{-1}AS = \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 8 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 2/3 - 8/3 & 4/3 - 4/3 \\ -1/3 + 16/3 & -2/3 + 8/3 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} -2 & 0 \\ 5 & 2 \end{pmatrix}}$$

$6/3 = 2$

6

2. Let $V \subseteq \mathbb{R}^3$ be the plane spanned by vectors $v_1 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto V. Answer the following questions. Show work and give reasons justifying your answers to get full credit.

(a) (4 points) Find an orthonormal basis $\{u_1, u_2\}$ for the image of T.

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{1+4+4}} v_1 = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

$$v_2' = v_2 - (v_2 \cdot u_1) u_1$$

$$= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix} \begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix}$$

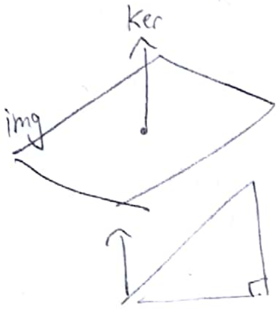
$$= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - 0$$

$$v_2 \cdot v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = 0 + 2 - 2 = 0 \quad 1$$

$$u_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\boxed{\text{basis: } \left\{ \begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix}, \begin{pmatrix} 0 \\ \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix} \right\}}$$

(b) (3 points) Find an orthonormal basis $\{u_3\}$ for the kernel of T.



mat of T: $\begin{pmatrix} 1/3 & 0 \\ 2/3 & 1/\sqrt{2} \\ -2/3 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 & -2/3 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

$= \begin{pmatrix} 1/9 \end{pmatrix}$

$\mathbb{R}$

$\mathbb{R}^3 \to \mathbb{R}^3$: $\ker(T) = \dfrac{v_1 \times v_2}{\text{mag}} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -2 \\ 0 & 1 & 1 \end{vmatrix} = \hat{i}(2+2) - \hat{j}(1) + \hat{k}(1)$

$= 4\hat{i} - \hat{j} + \hat{k}$

$\dfrac{1}{\sqrt{16+1+1}} \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$

$\dfrac{1}{\sqrt{18}} \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} \frac{2\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{6} \\ \frac{\sqrt{2}}{6} \end{pmatrix}}$

$3\sqrt{2}$

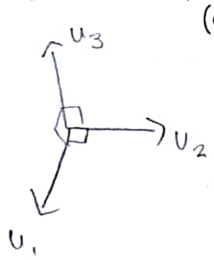
18: 9, 2

$3\sqrt{2}$

$\dfrac{\sqrt{2}}{6}$

3/

(c) (3 points) Let $R : \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection about the plane V and let M be the matrix of R with respect to the coordinates $\{u_1, u_2, u_3\}$. Find M



$$R : \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \to \begin{pmatrix} u_1 \\ u_2 \\ -u_3 \end{pmatrix}$$

$$\boxed{M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}$$

3. (10 points) Say whether each of the following statements is true or false.
+2 for each correct answer, +1 for no answer and -1 for each wrong answer!
No need to show work for this question. Also no partial credit will be given for this question. So double check your answers

(a) There exists an orthogonal transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ which is not invertible.

$\boxed{\text{False}}$

(b) There exists a 2×2 matrix A, a 2×1 matrix B and a 1×2 matrix C such that $A = BC$ and A is invertible.

$\boxed{\text{False}}$

(c) The matrices $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ are similar.

$\boxed{\text{True}}$

(d) There exists a 2×2 matrix such that it has the SAME subspace as kernel and image.

$\boxed{\text{True}}$

(e) There exists a 3×3 matrix such that it has the SAME subspace as kernel and image.

$\boxed{\text{False}}$

Space for scratch work (will not be evaluated):

ortho → presque length

$$\begin{pmatrix} - & - \\ - & - \end{pmatrix} = \begin{pmatrix} - \\ - \end{pmatrix}\begin{pmatrix} - & - \end{pmatrix}$$

$A A^T$ invertible if

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

$4-4$

$2-2$

$$\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} c & d \end{pmatrix} = \begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix}$$

$AS = SB$

ker + img = $n = 2$

$= 3$

det =

$acbd - adbc = 0$

no

$$A \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

4. Let $M = \begin{pmatrix} 4 & 25 \\ 0 & 0 \\ 3 & -25 \end{pmatrix}$. Answer the following questions. Show work to get full credit.

(a) (4 points) Find the QR factorization of M.

$u_1 = \frac{1}{\sqrt{16+9}} v_1 = \frac{1}{5}\begin{pmatrix}4\\0\\3\end{pmatrix}$

$v_2' = v_2 - (v_2 \cdot u_1) u_1$

$= \begin{pmatrix}25\\0\\-25\end{pmatrix} - \begin{pmatrix}25\\0\\-25\end{pmatrix} \cdot \begin{pmatrix}4/5\\0\\3/5\end{pmatrix} \begin{pmatrix}4/5\\0\\3/5\end{pmatrix}$

$= \begin{pmatrix}25\\0\\-25\end{pmatrix} - (20 - 15)\begin{pmatrix}4/5\\0\\3/5\end{pmatrix}$

$= \begin{pmatrix}25\\0\\-25\end{pmatrix} - \begin{pmatrix}4\\0\\3\end{pmatrix} = \begin{pmatrix}21\\0\\-28\end{pmatrix}$

$u_2 = \frac{v_2'}{\|v_2'\|} = \frac{v_2'}{\sqrt{1125}} = \frac{1}{15\sqrt{5}}\begin{pmatrix}21\\0\\-28\end{pmatrix}$

$$\boxed{Q = \begin{pmatrix} 4/5 & \frac{21}{15\sqrt{5}} \\ 0 & 0 \\ 3/5 & \frac{-28}{5\sqrt{5}} \end{pmatrix} \quad R = \begin{pmatrix} 5 & 5 \\ 0 & 5\sqrt{5} \end{pmatrix}}$$

21 × 21 = 21 + 420 = 441

16 + 6 = 22

28 × 28 = 224 + 560 = 784

784 + 441 = 1225

25) 1125 = 0045; 100; 125

$5 \cdot 3\sqrt{5}$

$15\sqrt{5}$

$v_2 =$

$u_2 = \frac{v_2 - \text{proj}}{\text{mag(top)}}$

$u_2 \text{mag(top)} + \text{proj}$

21 + 4

-28 + 3

(b) (4 points) Find ALL least square solutions x^* to the system $A\overline{x} = \overline{b}$ where

$$A = \begin{pmatrix} 4 & 25 \\ 0 & 0 \\ 3 & -25 \end{pmatrix} \text{ and } \overline{b} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$$

$A^T(\vec{b} - A\vec{x}^*) = 0$

$A^T\vec{b} = A^T A \vec{x}^*$

$A\vec{x}^* = (A^TA)^{-1}A^T\vec{b}$

$A^TA = \begin{pmatrix} 4 & 0 & 3 \\ 25 & 0 & -25 \end{pmatrix}\begin{pmatrix} 4 & 25 \\ 0 & 0 \\ 3 & -25 \end{pmatrix}$

$= \begin{pmatrix} 16+9 & 100-75 \\ 100-75 & 625+625 \end{pmatrix}$

$= \begin{pmatrix} 25 & 25 \\ 25 & 1250 \end{pmatrix}$

$A^Tb = \begin{pmatrix} 4 & 0 & 3 \\ 25 & 0 & -25 \end{pmatrix}\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{0}$

$A^TA\vec{x}^* = \vec{0}$

$\begin{pmatrix} 25 & 25 \\ 25 & 1250 \end{pmatrix}\vec{x}^* = 0$

$\left(\begin{array}{cc|c} 25 & 25 & 0 \\ 25 & 1250 & 0 \end{array}\right)$

$\boxed{\vec{x}^* = \vec{0}}$

(c) (2 points) Can you find a vector $\overline{v}$ such that $||A\overline{v} - \overline{b}|| < 2$? If yes, find it. If not, explain why.

No, the min $\overset{\text{magnitude}}{\text{distance}}$ (least square solution)

$= ||A\vec{0} - \vec{b}|| = 3$ which > 2

$= ||\vec{b}||$