Problem 1 (10 points in total)

Consider the map $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1+v_2 \\ v_1+v_2 \end{bmatrix}$$

for any vector $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ in $\mathbb{R}^3$.

1. (2 points) Write down the definition of linear transformation.

Solution:

A map T is a linear ~~tion~~ transformation if and only if,

$T(k\vec{v}) = kT(\vec{v})$ (for an arbitrary constant $k \in \mathbb{R}$)

AND

$T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$ (for all vectors $\vec{v}$ and $\vec{w} \in \mathbb{R}^m$)

2. (2 points) Show that the map T is a linear transformation.

Solution:

$$T(k\vec{v}) = T\left(k\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = T\left(\begin{bmatrix} kv_1 \\ kv_2 \\ kv_3 \end{bmatrix}\right) = \begin{bmatrix} kv_1 \\ kv_2 \end{bmatrix} = k\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$kT(\vec{v}) = k\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Thus, $kT(\vec{v}) = T(k\vec{v})$

$$T(\vec{v}+\vec{w}) = T\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}\right) = T\left(\begin{bmatrix} v_1+w_1 \\ v_2+w_2 \\ v_3+w_3 \end{bmatrix}\right) = \begin{bmatrix} v_1+w_1 \\ v_2+w_2 \end{bmatrix}$$

$$T(\vec{v}) + T(\vec{w}) = T\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) + T\left(\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}\right) = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} v_1+w_1 \\ v_2+w_2 \end{bmatrix}$$

Thus, $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$

3. (3 points) Write down the matrix A such that $T(v) = Av$ for all vectors v in $\mathbb{R}^3$.

Solution: A is a 2×3 matrix.

$A = \begin{bmatrix} T(e_1) & T(e_2) & T(e_3) \end{bmatrix}$,

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$T(\vec{v}) = A\vec{v} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \vec{v}$

4. (2 points) Compute the rank of A.

Solution: The rank of A is 2.

5. (1 point) Let B be any matrix of the same size as A. Can B have rank larger than A?

Solution: No, because $\text{rank}(B) \leq n$, where n is the number of rows in A and B.
Since $n = 2$, $\text{rank}(B) \leq 2$
Since $2 = \text{rank}(A)$, $\text{rank}(B) \leq \text{rank}(A)$

Problem 2 (10 points in total)

1. (6 points) Give an example of three linear transformations $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $U\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that the matrices that represent T and S commute, while the matrices that represent S and U do not commute. (Recall: we say that an $n \times m$ matrix A represents a linear transformation $T\colon \mathbb{R}^m \to \mathbb{R}^n$ if $T(v) = Av$ for all $v \in \mathbb{R}^m$.)

Solution:

$$T(\vec{v}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{v}$$

$$S(\vec{v}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{v}$$

$$U(\vec{v}) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \vec{v}$$

$$T(S(\vec{v})) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{v}$$

$$S(T(\vec{v})) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{v}$$

Thus $T(S(\vec{v})) = S(T(\vec{v}))$

$$S(U(\vec{v})) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \vec{v} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \vec{v}$$

$$U(S(\vec{v})) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \vec{v}$$

Thus $S(U(\vec{v})) \neq U(S(\vec{v}))$

2. (4 points in total) Let $A = \begin{bmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

- (2 points) Give a geometric interpretation of the transformations represented by A and B. (In words, or using a drawing.)

$\frac{\sqrt{2}}{2}$

Solution: $(x\,0.5)^2 + (-x\,0.5)^2 = 1$

$0.25x^2 + 0.25x^2 = 1$

$0.5x^2 = 1$

$x^2 = 2$

$x = \sqrt{2}$

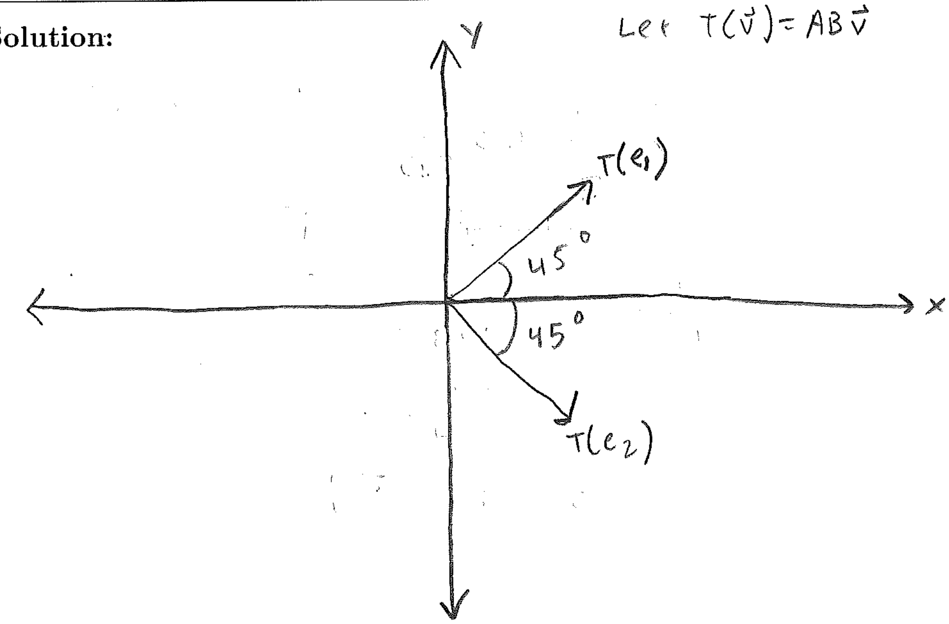
$\frac{\sqrt{2}}{2}$ $\frac{1}{\sqrt{2}}$

A is scaled by $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ and rotated counter-clockwise from the positive x axis by $\cos^{-1}(\frac{\sqrt{2}}{2}) = 45°$

B is a reflection over the X axis.

- (2 points) Draw the images of the standard basis unit vectors of $\mathbb{R}^2$ under the linear transformation represented by AB.

Solution: Let $T(\vec{v}) = AB\vec{v}$



Problem 3 (10 points in total)

Consider the following system of three linear equations in the variables x_1, x_2, x_3, x_4:

$$\begin{aligned} 2x_2 + x_4 &= 1 \\ x_1 + x_3 &= 1 \\ x_4 &= 1 \end{aligned}$$

1. (4 points) Solve the system using the Gauss-Jordan elimination algorithm.

Solution:

$$\left[\begin{array}{cccc|c} 0 & 2 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \rightarrow \text{swap } R_1, R_2$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \rightarrow \tfrac{1}{2}R_2 \rightarrow R_2$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \rightarrow R_2 - \tfrac{1}{2}R_3 \rightarrow R_3$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$x_1 + x_3 = 1$

$x_2 = 0$ Let $x_3 = t$

$x_4 = 1$ $x_1 = 1 - x_3 = 1 - t$

$$\vec{x} = \begin{bmatrix} 1-t \\ 0 \\ t \\ 1 \end{bmatrix}$$

2. (3 points) Let b_1, b_2, b_3 be arbitrary real numbers. How many solutions does the system

$$\begin{aligned} 2x_2 + x_4 &= b_1 \\ x_1 + x_3 &= b_2 \\ x_4 &= b_3 \end{aligned}$$

have?

Solution: $\begin{bmatrix} 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ in RREF is

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

rank = 3

m = # of col. = 4

Since rank = m + # of free vars, there must be 1 free variable and thus infinite solutions.

3. (3 points) Let A be any $n \times n$ matrix. Is there always a sequence of elementary row operations that transforms the identity matrix I_n into A? You should motivate your answer.

Solution: No; if $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ it is impossible to transform $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ to $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Problem 4 (10 points in total)

1. (2 points) Write down the definition of invertible matrix.

Solution: ~~matrix~~ $n \times n$ matrix A is invertible if and only if a unique $\vec{w} \in \mathbb{R}^n$ exists for every $\vec{v} \in \mathbb{R}^n$ in the transformation $T(\vec{v}) = \vec{w}$

It's RREF is I_n, given the matrix is $n \times n$.

2. (2 points) Give an example of a linear transformation $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ such that the matrix that represents T is not invertible.

Solution:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3. (4 points) Consider the following matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

Is A invertible? If yes, compute its inverse.

Solution: It's invertible

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 - 2R_1 \to R_2 \\ R_3 - 3R_1 \to R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -5 & -2 & 1 & 0 \\ 0 & -5 & -7 & -3 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_1 + 2R_2 \to R_1 \\ R_3 - 5R_2 \to R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -7 & -3 & 2 & 0 \\ 0 & -1 & -5 & -2 & 1 & 0 \\ 0 & 0 & 18 & 7 & -5 & 1 \end{array}\right] \quad -R_2 \to R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -7 & -3 & 2 & 0 \\ 0 & 1 & 5 & 2 & -1 & 0 \\ 0 & 0 & 18 & 7 & -5 & 1 \end{array}\right] \quad \tfrac{1}{18}R_3 \to R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -7 & -3 & 2 & 0 \\ 0 & 1 & 5 & 2 & -1 & 0 \\ 0 & 0 & 1 & 7/18 & -\frac{5}{18} & \frac{1}{18} \end{array}\right] \quad \begin{array}{l} R_1 + 3R_3 \to R_1 \\ R_2 - 5R_3 \to R_2 \end{array}$$

I_n, so it's invertible

$\downarrow$

$$\left[\begin{array}{ccc|} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right.$$

4. (2 points) Let A be an $n \times n$ matrix. Assume that A is not invertible. How many solutions does the system $Ax = b$ have?

Solution: There must be a $[0 \; 0 \; \ldots \; 0 \,|\, k]$ row, so infinite if $k = 0$ ~~oh~~ or none if $k \neq 0$

Problem 5 (10 points total; 2 points each)

Answer the following questions with true or false.

1. Any 4×3 matrix with rank equal to 3 is invertible.

   false

2. Let θ and η be any two angles with $\theta \neq \eta$. Let $T_\theta \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the counterclockwise rotation in $\mathbb{R}^2$ through θ, and $T_\eta \colon \mathbb{R}^2 \to \mathbb{R}^2$ the counterclockwise rotation in $\mathbb{R}^2$ through η. Then $T_\eta \circ T_\theta = T_\theta \circ T_\eta$.

   True

3. There exists a real number a for which the following matrix is in reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & a & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

   False

4. The transformation $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} v_1^2 + 2v_1 + 1 \\ v_1 + v_2 \end{bmatrix}$ is linear.

   False

5. A linear transformation $T \colon \mathbb{R}^n \to \mathbb{R}^n$ is invertible if and only if for every $v \in \mathbb{R}^n$ there exists a unique $w \in \mathbb{R}^n$ such that $T(v) = w$.

   True

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Use this sheet of paper if you need additional space.