Math 33A - Lectures 3 and 4
Fall 2018

Midterm 2

Instructions: You have 60 minutes to complete this exam. There are five questions, worth a total of 50 points. This test is closed book and closed notes. No calculator is allowed.

For full credit show all of your work legibly. Unless instructed otherwise, you need to justify your answers. Please write your solutions in the space below the questions; INDICATE if you go over the page and/or use the scrap pages at the end of this booklet.

Please take a moment to ensure that your booklet consists of ten pages, the last three being reserved for additional work.

Do not forget to write your full name, section and UID in the space below. For identification purposes, please sign below.

Full Name: Nathan Yuen
Student ID number: 305 100 413
Lecture: 4
Section: D

Signature: Nathan Yuen

Question	Points	Score
1	8	4
2	10	10
3	11	11
4	9	9
5	12	11
Total:	50	45

Problem 1.

For each of the following sentences, give an example of a matrix A with the following properties, or explain why it is impossible.

(a) [4pts.] A is a 3×3 matrix with $A^T A = -I_3$.

Not possible

$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $A^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$

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The middle entry of $A^T A$ is given by

$[b \; e \; h] \cdot \begin{bmatrix} b \\ e \\ h \end{bmatrix} = b^2 + e^2 + h^2$ ✓

which cannot equal negative one, the middle entry of $-I_3$.

Thus A does not exist

(b) [4pts.] A is a 2×2 matrix with integer entries and $\det(3A^2) = 75$.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 5/\sqrt{3} \end{bmatrix}$

$\det(3A^2) = 75$. Since A is 2×2,

$\det(3A^2) = 3^2 \det(A^2) = 9(\det A)^2 \rightarrow \det A = \pm \frac{5}{\sqrt{3}}$

Since $\det A = \frac{5}{\sqrt{3}}$ the equality

$9\left(\frac{5}{\sqrt{3}}\right)^2 = 75$

holds.

0

Problem 2.

(a) [5pts.] Find the least-squares solution $\vec{x}^*$ of the system

$$\begin{array}{cc} c_1 & c_2 \\ \end{array}$$
$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A} \vec{x} = \underbrace{\begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}}_{\vec{b}}. \qquad A^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

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Since $\vec{c}_1, \vec{c}_2$ linearly independent; $\ker A = 0$.

$$A^T A \vec{x}^* = A^T \vec{b}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\vec{x}^* = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\vec{x}^* = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 1 & 6 \\ 1 & 2 & 3 \end{array}\right] \xrightarrow{R_2 - \frac{1}{2}R_1} \left[\begin{array}{cc|c} 2 & 1 & 6 \\ 0 & \frac{3}{2} & 0 \end{array}\right] \begin{array}{l} R_1 - \frac{2}{3}R_2 \\ R_2 / (\frac{2}{3}) \end{array}$$

$$\rightarrow \left[\begin{array}{cc|c} 2 & 0 & 6 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1/2} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 0 \end{array}\right] \qquad \boxed{\vec{x}^* = \begin{bmatrix} 3 \\ 0 \end{bmatrix}}$$

(b) [5pts.] For the solution $\vec{x}^*$ you obtained in part (a), compute the error $||\vec{b} - A\vec{x}^*||$, where $\vec{b} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

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$$A\vec{x}^* = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$$

$$\vec{b} - A\vec{x}^* = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$$||\vec{b} - A\vec{x}^*|| = \sqrt{1+1+1} = \boxed{\sqrt{3}}$$

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Problem 3. *11pts.*

Find the QR-factorization of $M = \begin{array}{c} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \\ \vec{v}_1 \ \vec{v}_2 \end{array}$. Make sure to justify all steps.

$\vec{v}_1, \vec{v}_2$ are linearly independent.

$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{1+1+1}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{2}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Note: $\vec{v}_2 \cdot \vec{u}_1 = \frac{2}{\sqrt{3}}$

$= \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

$\vec{u}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|} = \frac{1}{\sqrt{\frac{4}{9}+\frac{1}{9}+\frac{1}{9}}} \left(\frac{1}{3}\right) \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{\frac{2}{3}}} \left(\frac{1}{3}\right) \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

Thus $Q = [\vec{u}_1, \vec{u}_2] = \begin{bmatrix} 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$

$R = \begin{bmatrix} \|\vec{v}_1\| & \vec{v}_2 \cdot \vec{u}_1 \\ 0 & \|\vec{w}_2\| \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 2/\sqrt{3} \\ 0 & \sqrt{2/3} \end{bmatrix}$

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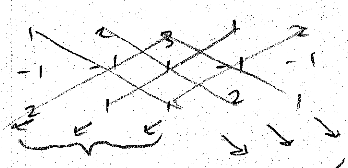
Problem 4.

Find the determinant of the following matrices. You can use any method you want, but make sure to justify each step.

(a) [4pts.] $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$.

Use Sarrus' rule.



$\det A = -1 + 4 - 3 - (-6 + 1 - 2)$

$= 0 + 7 = \boxed{7}$

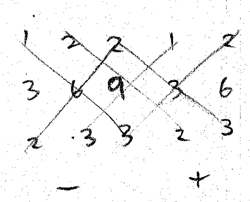
(b) [5pts.] $B = \begin{bmatrix} 2 & 3 & 1 & 0 \\ 1 & 0 & 2 & 2 \\ 3 & 5 & 6 & 9 \\ 2 & 0 & 3 & 3 \end{bmatrix}$.

$\begin{bmatrix} + & - & & \\ - & + & & \\ + & - & & \\ - & + & & \end{bmatrix}$

Use Laplace Expansion on the 2nd column.

$\det B = -3 \det \begin{bmatrix} 1 & 2 & 2 \\ 3 & 6 & 9 \\ 2 & 3 & 3 \end{bmatrix} + 0 - 5 \det \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 2 \\ 2 & 3 & 3 \end{bmatrix} + 0$

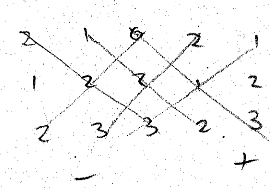
Sarrus' rule:



$18 + 36 + 18 - (24 + 27 + 18)$

$= 3$

Sarrus' rule:



$12 + 4 + 0 - (0 + 12 + 3)$

$= 1$

$\det B = -3(3) - 5(1) = \boxed{-14}$

3

Problem 5.

Let $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Let P be the 4-parallelepiped defined by the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$.

(a) [6pts.] Find the 4-volume of P.

Let $A = \begin{bmatrix} | & | & | & | \\ \vec{v}_4, & \vec{v}_3, & \vec{v}_2, & \vec{v}_1 \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

Then $\text{Vol}(P) = \det A$, since we are in $\mathbb{R}^4$ and A is 4×4.

Since A is triangular $\det A =$ product along the diagonal.

$\text{Vol } P = \det A = (1)(1)(1)(1) = \boxed{1}$ ✓ 6/6

[This problem continues from the previous page.]

Recall that $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

(b) [6pts.] Consider the linear transformation $T(\vec{x}) = \begin{bmatrix} 2 & 4 & 0 & 1 \\ 1 & -3 & -2 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 2 & -5 \end{bmatrix} \vec{x}$.

Find the 4-volume of the 4-parallelepiped defined by the vectors $T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)$ and $T(\vec{v}_4)$.

Let Q be our desired parallelipiped, A be the matrix of T.

Vol Q = (Vol P) |det A|. — 1

Since Vol P = 1, Vol Q = |det A|

A is a block matrix so its determinant is the product of the upper-left and lower-right 2x2 matrices that make up 4x4 A

$\text{Vol } Q = |\det A| = \left| \det \begin{bmatrix} 2 & 4 \\ 1 & -3 \end{bmatrix} * \det \begin{bmatrix} -1 & 3 \\ 2 & -5 \end{bmatrix} \right|$

$= \left| ((-3)(2) - (4)(1))\,((-1)(-5) - (3)(2)) \right|$

$= |(-10)(-1)|$

$= \boxed{10}$

5/6

4

Use this page for any additional work. Make sure to clearly state which problem you are solving on this page.

Scratch work only

$3 \det A \det A = 75$

$\det A = 5$ or -5

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & & \\ & & \\ & & \end{bmatrix}$$

$$\begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix} \qquad d^2 + b^2 + f^2 = 0$$

$$\begin{bmatrix} 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 2/\sqrt{3} \\ 0 & \sqrt{2/3} \end{bmatrix}$$

$-\frac{2}{\sqrt{6}} \quad \frac{\sqrt{2}}{\sqrt{3}}$

$-\cancel{2} + 1 + 1$

$\cancel{4} + 8$

1

$-\frac{2\sqrt{2}}{\sqrt{2}\sqrt{3}\sqrt{3}}$

1

$2+$

$-\frac{2}{3}$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\frac{2}{3} \quad -\frac{2}{3}$

Use this page for any additional work. Make sure to clearly state which problem you are solving on this page.

Scratch work only

$$\begin{matrix} 2 & 4 & 0 & 1 \\ 1 & -3 & -2 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 2 & -5 \end{matrix}$$

$$-1 \begin{bmatrix} 2 & 4 & 1 \\ 1 & -3 & 0 \\ 0 & 0 & -5 \end{bmatrix} - 3 \begin{bmatrix} 2 & 4 & 0 \\ 1 & -3 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$(-5)(-1) \det \begin{bmatrix} 2 & 4 \\ 1 & -3 \end{bmatrix}$$

$$+ (-3)(2) \det \begin{bmatrix} 2 & 4 \\ 1 & -3 \end{bmatrix}$$

$$(-10)(5-6)$$

$$= 10$$

1

25

Use this page for any additional work. Make sure to clearly state which problem you are solving on this page.