Problem 1.

For each of the following sentences, give an example of a matrix A with the following properties, or explain why it is impossible.

(a) [4pts.] A is a 3×3 matrix with $A^T A = -I_3$.

Not possible: Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \rightarrow A^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$

The first term in the (1,1) position of $-I_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ is -1

The first term in the (1,1) pos of A^TA can be calculated as the dot product of the first row of A^T w/ the first column of A

$(a,b,c)\cdot(a,b,c) = a^2+b^2+c^2$ 4/4

$a^2+b^2+c^2 \geq 0 \therefore a^2+b^2+c^2 \neq -1 \rightarrow A^TA$ can never equal

(b) [4pts.] A is a 2×2 matrix with integer entries and $\det(3A^2) = 75$.

$\det(3A^2) = \det(A^2)\cdot 3^2 = 9\det(A)^2 = 75$

$\det(A)^2 = \frac{25}{3}$

$\det(A) = \frac{5}{\sqrt{3}}$

Not possible; determinant of a matrix with all integer entries must also be integer.

4

Problem 2.

✓ (a) [5pts.] Find the least-squares solution $\vec{x}^*$ of the system

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}.$$

5

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\vec{x}^* = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\vec{x}^* = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 1 & 6 \\ 1 & 2 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1/2 & 3 \\ 0 & 3/2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 0 \end{array}\right].$$

$$x = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

✓ (b) [5pts.] For the solution $\vec{x}^*$ you obtained in part (a), compute the error $||\vec{b} - A\vec{x}^*||$, where $\vec{b} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

5

$$\left\|\begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix}\right\| = \left\|\begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}\right\| = \left\|\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}\right\|$$

$$\text{error} = \sqrt{3}$$

Problem 3. *11pts.*

Find the QR-factorization of $M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$. Make sure to justify all steps.

Gram-Schmidt to get Q, R

(1) unitize v_1

$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$ $\quad \sqrt{3} = R_{1,1}$

(2) w_2 s.t $w_2 \perp u_1$

$w_2 = v_2 - (v_2 \cdot u_1)u_1 = \begin{bmatrix}0\\1\\1\end{bmatrix} - \left(\frac{2}{\sqrt{3}}\right)\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\1\\1\end{bmatrix} - \frac{2}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}-2/3\\1/3\\1/3\end{bmatrix}$ $\quad \frac{2}{\sqrt{3}} = R_{1,2}$

(3) unitize w_2

$u_2 = \frac{w_2}{\|w_2\|} = \frac{1}{\sqrt{\frac{6}{9}}}\begin{bmatrix}-2/3\\1/3\\1/3\end{bmatrix} = \frac{3}{\sqrt{6}}\begin{bmatrix}-2/3\\1/3\\1/3\end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix}-2\\1\\1\end{bmatrix}$ $\quad \frac{\sqrt{6}}{3} = R_{2,2}$

$Q = [u_1, u_2] \quad R = \begin{bmatrix}\|v_1\| & v_2 \cdot u_1 \\ 0 & \|w_2\|\end{bmatrix}$

$Q = \begin{bmatrix}1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6}\end{bmatrix} \quad R = \begin{bmatrix}\sqrt{3} & 2/\sqrt{3} \\ 0 & \sqrt{6}/3\end{bmatrix}$

11

Problem 4.

Find the determinant of the following matrices. You can use any method you want, but make sure to justify each step.

✓ (a) [4pts.] $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$.   set R_1, laplace expand

$$1 \det \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} - 2 \det \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix} + 3 \det \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$1(-2) - 2(-3) + 3(1) = -2 + 6 + 3$$

$$\det(A) = 7$$

(b) [5pts.] $B = \begin{bmatrix} 2 & 3 & 1 & 0 \\ 1 & 0 & 2 & 2 \\ 3 & 5 & 6 & 9 \\ 2 & 0 & 3 & 3 \end{bmatrix}$.   Row opr / Gaussian elim

$$\cancel{\begin{bmatrix} 2 & 3 & 1 & 0 \\ 1 & 0 & 2 & 2 \\ 3 & 5 & 6 & 9 \\ 2 & 0 & 3 & 3 \end{bmatrix} \xrightarrow{(-1)} \begin{bmatrix} 1 & 0 & 2 & 2 \\ 2 & 3 & 1 & 0 \\ 3 & 5 & 6 & 9 \\ 2 & 0 & 3 & 3 \end{bmatrix} \xrightarrow{(-1)} \begin{bmatrix} 1 & 0 & 2 & 2 \\ 2 & 0 & 3 & 3 \\ 3 & 5 & 6 & 9 \\ 2 & 3 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 3 \\ 0 & 3 & -3 & -4 \end{bmatrix} \xrightarrow{(-1)} \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 3 & -3 & -4 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}}$$

$$\cancel{\det(B)(-1)(-1)(-1) = -9}$$

$$\cancel{\det(B) = 9}$$

$$\begin{bmatrix} 2 & 3 & 1 & 0 \\ 1 & 0 & 2 & 2 \\ 3 & 5 & 6 & 9 \\ 2 & 0 & 3 & 3 \end{bmatrix} \xrightarrow{(-1)} \begin{bmatrix} 1 & 0 & 2 & 2 \\ 2 & 3 & 1 & 0 \\ 3 & 5 & 6 & 9 \\ 2 & 0 & 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 3 & -3 & -4 \\ 0 & 5 & 0 & 3 \\ 0 & 0 & -1 & -1 \end{bmatrix} \xrightarrow{(\frac{1}{3})} \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & -1 & -4/3 \\ 0 & 0 & 5 & 29/3 \\ 0 & 0 & -1 & -1 \end{bmatrix} \xrightarrow{(\frac{1}{5})} \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & -1 & -4/3 \\ 0 & 0 & 1 & 29/15 \\ 0 & 0 & 0 & 14/15 \end{bmatrix}$$

$$\det(B)(-1)\left(\frac{1}{3}\right)\left(\frac{1}{5}\right) = \frac{14}{15}$$

$$3 + \frac{20}{3} \qquad \frac{29}{15}$$

$$\det(B) = -14$$

$$\frac{9}{3} \quad \frac{29}{3}$$

Problem 5.

Let $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Let P be the 4-parallelepiped defined by the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$.

(a) [6pts.] Find the 4-volume of P.

-1

$$\mathrm{Vol}(P) = \left| \det \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \right| = (-1)\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = (-1)(-1)\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

✓

$$\mathrm{Vol}(P) = 1$$

5/6

[This problem continues from the previous page.]

Recall that $\vec{v}_1 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}$ and $\vec{v}_4 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$.

(b) [6pts.] Consider the linear transformation $T(\vec{x}) = \begin{bmatrix} 2 & 4 & 0 & 1 \\ 1 & -3 & -2 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 2 & -5 \end{bmatrix} \vec{x}$.

Find the 4-volume of the 4-parallelepiped defined by the vectors $T(\vec{v}_1)$, $T(\vec{v}_2)$, $T(\vec{v}_3)$ and $T(\vec{v}_4)$.

-1

$$\mathrm{Vol}(P) \left| \det\left(\begin{bmatrix} 2 & 4 & 0 & 1 \\ 1 & -3 & -2 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 2 & -5 \end{bmatrix}\right) \right| = \mathrm{Vol}(T(P))$$

$$\det\left(\begin{bmatrix} 2 & 4 & 0 & 1 \\ 1 & -3 & -2 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 2 & -5 \end{bmatrix}\right) = -0\det() + 0\det() - 2\det\begin{bmatrix} 2 & 4 & 1 \\ 1 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - 5\det\begin{bmatrix} 2 & 4 & 0 \\ 1 & -3 & -2 \\ 0 & 0 & -1 \end{bmatrix}$$

$-1 \quad -1$

$$-2\det\begin{bmatrix} 2 & 4 & 1 \\ 1 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - 5\det\begin{bmatrix} 2 & 4 & 0 \\ 1 & -3 & -2 \\ 0 & 0 & -1 \end{bmatrix}$$

$$-2(3(-6-4)) - 5\det(-1(-6-4))$$

$$60 - 50$$

$$\det[A] = 10$$

5/6

$$\mathrm{Vol}(P)(10) = \mathrm{Vol}(T(P))$$

$$1(10) = \mathrm{Vol}(T(P))$$

✓

$$\mathrm{Vol}(T(P)) = 10$$

Scanned by CamScanner