Problem 1.

For each of the following sentences, give an example of a matrix A with the following properties, or explain why it is impossible.

(a) [4pts.] A is a 3×3 matrix with $A^T A = -I_3$.

Solution: This is impossible, because taking determinants yields

$$-1 = (-1)^3 = \det(-I_3) = \det(A^T A) = \det(A^T)\det(A) = \det(A)^2.$$

(b) [4pts.] A is a 2×2 matrix with integer entries and $\det(3A^2) = 75$.

Solution: This is also impossible. Indeed, for any 2×2 matrix B we have $\det(3B) = 3^2 \det(B)$ because to get $3B$ from B we multiplied *both* rows of B by 3. Thus, we have

$$75 = \det(3A^2) = 3^2 \det(A^2) = 9\det(A)^2$$

and solving for $\det(A)$ yields that $\det(A) = \frac{5}{\sqrt{3}}$. But a matrix with integer entries must have integer determinant, so this can't be.

Problem 2.

(a) [5pts.] Find the least-squares solution $\vec{x}^*$ of the system

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}.$$

Solution: Letting $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$, the formula to solve via least squares an inconsistent system is $A^T A\vec{x} = A^T\vec{b}$. We compute

$$A^T\vec{b} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4+2 \\ 4-1 \end{bmatrix} = \begin{bmatrix} 6 \end{bmatrix} 3$$

and

$$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

so that we need to solve the system

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \vec{x}^* = \begin{bmatrix} 6 \\ 3 \end{bmatrix}.$$

Inverting the matrix $A^T A$ yields

$$\vec{x}^* = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \frac{1}{2 \cdot 2 - 1 \cdot 1} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 6 \end{bmatrix} 3 =$$

$$= \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \cdot 6 - \frac{1}{3} \cdot 3 \\ -\frac{1}{3} \cdot 6 + \frac{2}{3} \cdot 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$

(b) [5pts.] For the solution $\vec{x}^*$ you obtained in part (a), compute the error $||\vec{b} - A\vec{x}^*||$, where $\vec{b} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Solution: We compute

$$A\vec{x}^* = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$$

and thus

$$||\vec{b} - A\vec{x}^{*}|| = || \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} || = || \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} || = \sqrt{1+1+1} = \sqrt{3}.$$

Problem 3. *11pts.*

Find the QR-factorization of $M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$. Make sure to justify all steps.

Solution: We start by doing Gram-Schmidt on the columns of M, $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, which are non-parallel and thus linearly independent.

We get

$$\vec{u}_1 = \frac{1}{||\vec{v}_1||}\vec{v}_1 = \frac{1}{\sqrt{1+1+1}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Then,

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\,\vec{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{1+1}{\sqrt{3}} \right) \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}.$$

Finally,

$$\vec{u}_2 = \frac{1}{||\vec{w}_2||}\vec{w}_2 = \frac{1}{\sqrt{\frac{4}{9}+\frac{1}{9}+\frac{1}{9}}} \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \frac{1}{\sqrt{\frac{2}{3}}} \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = \sqrt{\frac{3}{2}} \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}.$$

We can now write down the matrices Q and R such that $M = QR$:

$$Q = [\vec{u}_1, \vec{u}_2] = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{3}\sqrt{\frac{3}{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{3}\sqrt{\frac{3}{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{3}\sqrt{\frac{3}{2}} \end{bmatrix} \text{ and } R = \begin{bmatrix} ||\vec{v}_1|| & \vec{v}_2 \cdot \vec{u}_1 \\ 0 & ||\vec{w}_2|| \end{bmatrix} = \begin{bmatrix} \sqrt{3} & \frac{2}{\sqrt{3}} \\ 0 & \sqrt{\frac{2}{3}} \end{bmatrix}.$$

Problem 4.

Find the determinant of the following matrices. You can use any method you want, but make sure to justify each step.

(a) [4pts.] $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$.

Solution: We use Sarrus's rule: we write

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{matrix} 1 & 2 \\ -1 & -1 \\ 2 & 1 \end{matrix}$$

and then sum the products on the three top-left-to-bottom-right diagonals and subtract the products on the three bottom-left-to-top-right diagonals:

$$\det A = 1 \cdot (-1) \cdot 1 + 2 \cdot 1 \cdot 2 + 3 \cdot (-1) \cdot 1 - 2 \cdot (-1) \cdot 3 - 1 \cdot 1 \cdot 1 - 1 \cdot (-1) \cdot 2 =$$

$$= -1 + 4 - 3 + 6 - 1 + 2 = 7.$$

(b) [5pts.] $B = \begin{bmatrix} 2 & 3 & 1 & 0 \\ 1 & 0 & 2 & 2 \\ 3 & 5 & 6 & 9 \\ 2 & 0 & 3 & 3 \end{bmatrix}$.

Solution: The most efficient method to compute this determinant is by using Laplace expansion on the second column, which contains two zeros. We obtain

$$\det B = -3 \cdot \det \begin{bmatrix} 1 & 2 & 2 \\ 3 & 6 & 9 \\ 2 & 3 & 3 \end{bmatrix} - 5 \cdot \det \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 2 \\ 2 & 3 & 3 \end{bmatrix}.$$

We then use Sarrus's rule for each of the 3×3 matrices in the above equation. For the first one, we write

$$\begin{bmatrix} 1 & 2 & 2 \\ 3 & 6 & 9 \\ 2 & 3 & 3 \end{bmatrix} \begin{matrix} 1 & 2 \\ 3 & 6 \\ 2 & 3 \end{matrix}$$

and then sum the products on the three top-left-to-bottom-right diagonals and subtract the products on the three bottom-left-to-top-right diagonals:

$$\det \begin{bmatrix} 1 & 2 & 2 \\ 3 & 6 & 9 \\ 2 & 3 & 3 \end{bmatrix} = 1 \cdot 6 \cdot 3 + 2 \cdot 9 \cdot 2 + 2 \cdot 3 \cdot 3 - 2 \cdot 6 \cdot 2 - 3 \cdot 9 \cdot 1 - 3 \cdot 3 \cdot 2 =$$

$$= 18 + 36 + 18 - 24 - 27 - 18 = 3.$$

For the second one, we write

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 2 \\ 2 & 3 & 3 \end{bmatrix} \begin{matrix} 2 & 1 \\ 1 & 2 \\ 2 & 3 \end{matrix}$$

and then sum the products on the three top-left-to-bottom-right diagonals and subtract the products on the three bottom-left-to-top-right diagonals:

$$\det \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 2 \\ 2 & 3 & 3 \end{bmatrix} = 2 \cdot 2 \cdot 3 + 1 \cdot 2 \cdot 2 + 0 \cdot 1 \cdot 3 - 2 \cdot 2 \cdot 0 - 3 \cdot 2 \cdot 2 - 3 \cdot 1 \cdot 1 =$$

$$= 12 + 4 + 0 - 0 - 12 - 3 = 1.$$

We conclude that

$$\det B = -3 \cdot \det \begin{bmatrix} 1 & 2 & 2 \\ 3 & 6 & 9 \\ 2 & 3 & 3 \end{bmatrix} - 5 \cdot \det \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 2 \\ 2 & 3 & 3 \end{bmatrix} = -3 \cdot 3 - 5 \cdot 1 = -9 - 5 = -14.$$

Problem 5.

Let $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Let P be the 4-parallelepiped defined by the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$.

(a) [6pts.] Find the 4-volume of P.

Solution: First, notice that the 4-parallelepiped defined by the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ and $\vec{v}_4$ is the same as the 4-parallelepiped defined by the vectors $\vec{v}_4, \vec{v}_3, \vec{v}_2$ and $\vec{v}_1$. This is obvious from our geometric intuition, and it is also clear by looking at the definition 6.3.5 (of an m-parallelepiped) in the textbook.

Next, we use the formula in theorem 6.3.6, in the special case that $m = n$ since both numbers are equal to 4 in this case. We have

$$\operatorname{Vol}(P) = \operatorname{Vol}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4) = \operatorname{Vol}(\vec{v}_4, \vec{v}_3, \vec{v}_2, \vec{v}_1) = \left|\det \begin{bmatrix} \vec{v}_4 & \vec{v}_3 & \vec{v}_2 & \vec{v}_1 \end{bmatrix}\right| =$$

$$= \left|\det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}\right| = |1| = 1,$$

where the last computation of the determinant is immediate since the matrix $\begin{bmatrix} \vec{v}_4 & \vec{v}_3 & \vec{v}_2 & \vec{v}_1 \end{bmatrix}$ is lower triangular (this is why we swapped the orders of the vectors at the beginning of the problem).

(b) [6pts.] Consider the linear transformation $T(\vec{x}) = \begin{bmatrix} 2 & 4 & 0 & 1 \\ 1 & -3 & -2 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 2 & -5 \end{bmatrix} \vec{x}$.

Find the 4-volume of the 4-parallelepiped defined by the vectors $T(\vec{v}_1)$, $T(\vec{v}_2)$, $T(\vec{v}_3)$ and $T(\vec{v}_4)$.

Solution: Denote $A = \begin{bmatrix} 2 & 4 & 0 & 1 \\ 1 & -3 & -2 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 2 & -5 \end{bmatrix}$. Then theorem 6.3.7 gives that the volume we are asked to find is

$$\operatorname{Vol}(A\vec{v}_1, A\vec{v}_2, A\vec{v}_3, A\vec{v}_4) = |\det A| \cdot \operatorname{Vol}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4) = |\det A| \cdot 1 = |\det A|,$$

so all we need to do is compute the determinant of the matrix A.

Since this matrix is very close to being upper triangular, we use row operations.

We have

$$\begin{bmatrix} 2 & 4 & 0 & 1 \\ 1 & -3 & -2 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 2 & -5 \end{bmatrix} \xrightarrow[R_4+2R_3]{R_2-\frac{1}{2}R_1} \begin{bmatrix} 2 & 4 & 0 & 1 \\ 0 & -5 & -2 & -\frac{1}{2} \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the determinant of the latter matrix is $2 \cdot (-5) \cdot (-1) \cdot 1 = 10$. The row operations we have performed do not change the determinant, so we also have $\det A = 10$, and we conclude that $\mathrm{Vol}\,(A\vec{v}_1, A\vec{v}_2, A\vec{v}_3, A\vec{v}_4) = |10| = 10$.