Math 33A - Lectures 3 and 4
Fall 2018

Midterm 1

Instructions: You have 60 minutes to complete this exam. There are five questions, worth a total of 50 points. This test is closed book and closed notes. No calculator is allowed.

For full credit show all of your work legibly. Unless instructed otherwise, you need to justify your answers. Please write your solutions in the space below the questions; INDICATE if you go over the page and/or use the scrap pages at the end of this booklet.

Please take a moment to ensure that your booklet consists of ten pages, the last three being reserved for additional work.

Do not forget to write your full name, section and UID in the space below. For identification purposes, please sign below.

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Lecture: 4

Section: D

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Question	Points	Score
1	8	8
2	10	10
3	11	10
4	10	7
5	11	6
Total:	50	41

Problem 1.

For each of the following sentences, give an example of a matrix A with the following properties, or explain why it is impossible.

(a) [4pts.] A is a 3×6 matrix with rank and nullity both equal to 3.

$$A : \mathbb{R}^6 \to \mathbb{R}^3$$

4

$$\boxed{A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}}$$

$$\operatorname{im} A = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \mathbb{R}^3$$

$$\operatorname{rank} A = \dim(\operatorname{im} A) = 3$$

$$\ker A = \operatorname{span}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

This is a basis for ker A since the vectors are linearly independent.

$$\operatorname{nullity} A = \dim(\ker A) = 3$$

(b) [4pts.] A is a 6×3 matrix with rank and nullity both equal to 3.

$\boxed{\text{Does Not Exist}}$

$$A : \mathbb{R}^3 \to \mathbb{R}^6$$

4

By Rank-Nullity Theorem:

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = \dim(\mathbb{R}^3)$$

domain of $A = \mathbb{R}^3$

$$\begin{array}{ccccc} \downarrow & & \downarrow & & \downarrow \\ 3 & + & 3 & = & 3 \end{array}$$

which is clearly false so A does not exist.

Problem 2.

Recall that two vectors $\vec{v}, \vec{w} \in \mathbf{R}^n$ are perpendicular if their dot product is zero: $\vec{v} \cdot \vec{w} = \vec{0}$.

(a) [5pts.] Find a nonzero matrix A such that $A\vec{x}$ is perpendicular to the vector $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ for every $\vec{x} \in \mathbf{R}^3$. [You do not need to justify how you found A, but you do need to show that your choice of A satisfies the prescribed condition.]

$x : (x_1 \quad x_2 \quad x_3)$

$$\boxed{A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}} \qquad A\vec{x} = \begin{bmatrix} x_2 \\ -x_1 \\ x_1 - x_2 \end{bmatrix}$$

$$A\vec{x} \cdot \vec{v} = (x_2)(1) + (-x_1)(1) + (x_1 - x_2)(1)$$

$$= x_2 - x_1 + x_1 - x_2 = 0$$

5

(b) [5pts.] For the matrix A you found in part (a), let $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ be the corresponding linear transformation. Find, with justification, a basis of the image of T.

The basis of the image for A is the span of the columns of A containing pivots.

$$RREF(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ so $\vec{c}_1, \vec{c}_2$ contain pivots and form the basis for A.

$$\left(A = \begin{bmatrix} | & | & | \\ c_1 & c_2 & c_3 \\ | & | & | \end{bmatrix} \right)$$

$$\text{Basis}: \boxed{\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}}$$

5

Problem 3.

Let $\vec{x}, \vec{y}$ be two nonzero vectors in $\mathbf{R}^n$. Consider the set

$$V = \{\vec{v} \in \mathbf{R}^n \text{ such that } \vec{v} \cdot \vec{x} = \vec{v} \cdot \vec{y}\}.$$

(a) [3pts.] Prove that V is a subspace of $\mathbf{R}^n$.

Must satisfy:

1) Closed under addition

$\vec{v}, \vec{w} \in V$, prove $\vec{v} + \vec{w} \in V$

$\vec{v} \cdot \vec{x} = \vec{v} \cdot \vec{y} \rightarrow v_1 x_1 + v_2 x_2 + \dots v_n x_n = v_1 y_1 + v_2 y_2 + \dots v_n y_n$

$\vec{w} \cdot \vec{x} = \vec{w} \cdot \vec{y} \rightarrow w_1 x_1 + w_2 x_2 + \dots w_n x_n = w_1 y_1 + w_2 y_2 + \dots w_n y_n$

2/3

Add these equations together:

$(v_1 + w_1) x_1 + (v_2 + w_2) x_2 + \dots (v_n + w_n) x_n = (v_1 + w_1) y_1 + (v_2 + w_2) y_2 + (v_n + w_n) y_n$

$\rightarrow (\vec{v} + \vec{w}) \cdot \vec{x} = (\vec{v} + \vec{w}) \cdot \vec{y} \rightarrow \vec{v} + \vec{w} \in V$ ✓

cont. on back paper.

(b) [3pts.] Let now $\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$ be vectors in $\mathbf{R}^4$ and let V be defined as above.

Show that $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$ belong to V.

$\vec{v}_1 \cdot \vec{x} = (2)(1) + (1)(1) + (0)(-1) + (1)(1) = 4$

$\vec{v}_1 \cdot \vec{y} = (1)(1) + (2)(1) + (-1)(-1) + (0)(1) = 4$

$\vec{v}_1 \cdot \vec{x} = \vec{v}_1 \cdot \vec{y} \rightarrow \vec{v}_1 \in V$

3/3

$\vec{v}_2 \cdot \vec{x} = (2)(0) + (1)(1) + (0)(1) + (1)(0) = 1$

$\vec{v}_2 \cdot \vec{y} = (1)(0) + (2)(1) + (-1)(1) + (0)(0) = 1$

$\vec{v}_2 \cdot \vec{x} = \vec{v}_2 \cdot \vec{y} \rightarrow \vec{v}_2 \in V$

$\vec{v}_3 \cdot \vec{x} = (2)(1) + (1)(-1) + (0)(-1) + (1)(-1) = 0$

$\vec{v}_3 \cdot \vec{y} = (1)(1) + (2)(-1) + (-1)(-1) + (0)(-1) = 0$

$\vec{v}_3 \cdot \vec{x} = \vec{v}_3 \cdot \vec{y} \rightarrow \vec{v}_3 \in V$

This problem continues from the previous page. Recall that $\vec{x}, \vec{y}, \vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ are defined in question (b), and $V = \{\vec{v} \in \mathbf{R}^4 \text{ such that } \vec{v} \cdot \vec{x} = \vec{v} \cdot \vec{y}\}$.

(c) [5pts.] Prove that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis of V.

First, let's show $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent.

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}, \text{ prove } c_1 = c_2 = c_3 = 0$$

In matrix form:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 + R_1 \\ R_4 - R_1 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] R_3 - R_2$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] R_3/(-2) \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \begin{array}{l} R_1 - R_3 \\ R_2 + R_3 \\ \\ R_4 + 2R_3 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow c_1 = c_2 = c_3 = 0 \text{ so}$$

$\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent. ✓

$\dim(V) = 4 \longleftrightarrow V = \mathbb{R}^4$ since $V \leq \mathbb{R}^4$

However, $\vec{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \notin V \rightarrow V < \mathbb{R}^4 \rightarrow \dim V < 4$.

But $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in V$ and are linearly dependent, so span$(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ has dimension 3 and $\in V$. Clever!

Thus $\dim V \geq 3$.

Since $3 \leq \dim V < 4$, $\dim V = 3$. 5/5

3 linearly independent vectors spanning a subspace with dimension 3 form a basis for the subspace.

Basis for V: $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

Problem 4.

(a) [5pts.] Using row-reduction find, if it exists, the inverse of the matrix

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 7 & 1 & 0 & 0 \\ 2 & 5 & 8 & 0 & 1 & 0 \\ 3 & 6 & 9 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 4 & 7 & 1 & 0 & 0 \\ 0 & -3 & -6 & -2 & 1 & 0 \\ 0 & -6 & -12 & -3 & 0 & 1 \end{array}\right] R_2/(-3)$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 4 & 7 & 1 & 0 & 0 \\ 0 & 1 & 2 & -\frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & -6 & -12 & -3 & 0 & 1 \end{array}\right]$$

At this point we can see that $R^3 = -6R^2$ on the left-hand side of the matrix. This means that rank $(A) = 2$. Since A is a 3×3 matrix, it can only be invertible if its rank is 3. $\boxed{A^{-1} \text{ Does Not Exist}}$ ✓

(b) [5pts.] Let A be the matrix defined in (a). Find all solutions $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ of the system

$$A\vec{x} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$ [There could be none.]

$$A\vec{x} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \longrightarrow \left[\begin{array}{ccc|c} 1 & 4 & 7 & -1 \\ 2 & 5 & 8 & 0 \\ 3 & 6 & 9 & 1 \end{array}\right] \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 4 & 7 & -1 \\ 0 & -3 & -6 & 2 \\ 0 & -6 & -12 & 4 \end{array}\right] R_3 + 2R_2 \rightarrow \left[\begin{array}{ccc|c} 1 & 4 & 7 & -1 \\ 0 & -3 & -6 & 2 \\ 0 & 0 & 0 & \cancel{8} \end{array}\right]$$

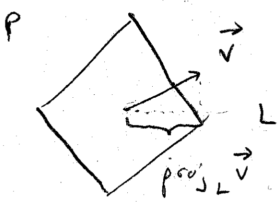
The bottom row is equivalent to $0x + 0y + 0z = 8$. This is untrue, so there are $\boxed{\text{no solutions}}$ and the system is inconsistent.

7

Problem 5.

Let P be the plane in $\mathbf{R}^3$ given by the equation $x - y + z = 0$.

(a) [5pts.] Let $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ be the reflection across the plane P. Find the matrix of T with respect to the standard basis of $\mathbf{R}^3$.



Let L = line where $L \perp P$. Here,

L = line spanned by $\vec{l} = [1 \quad -1 \quad 1]$.

3/5

$$\text{refl}_P \vec{v} = \vec{v} - 2\,\text{proj}_L \vec{v}$$

T with matrix A, let B = matrix for $-2\,\text{proj}_L \vec{v}$

$$\text{proj}_L \vec{v} = \frac{\vec{v}\cdot\vec{l}}{\vec{l}\cdot\vec{l}}\,\vec{l} = \frac{v_1 - v_2 + v_3}{3}\begin{bmatrix}1\\-1\\1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}v_1 - v_2 + v_3\\-v_1 + v_2 - v_3\\v_1 - v_2 + v_3\end{bmatrix}$$

$$B\vec{v} = -\frac{2}{3}\begin{bmatrix}1 & -1 & 1\\-1 & 1 & -1\\1 & -1 & 1\end{bmatrix}$$

$$A\vec{v} = I\vec{v} - B\vec{v} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} + \begin{bmatrix}\frac{2}{3} & -\frac{2}{3} & \frac{2}{3}\\-\frac{2}{3} & \frac{2}{3} & -\frac{2}{3}\\\frac{2}{3} & -\frac{2}{3} & \frac{2}{3}\end{bmatrix} = \boxed{\begin{bmatrix}\frac{5}{3} & -\frac{2}{3} & \frac{2}{3}\\-\frac{2}{3} & \frac{5}{3} & -\frac{2}{3}\\\frac{2}{3} & -\frac{2}{3} & \frac{5}{3}\end{bmatrix}}$$

(b) [6pts.] Find a basis $\mathfrak{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of $\mathbf{R}^3$ such that the $\mathfrak{B}$-matrix of T is diagonal. Write down the $\mathfrak{B}$-matrix of T.

$$\mathfrak{B} = \left\{ \underbrace{\begin{bmatrix}0\\1\\1\end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix}1\\1\\0\end{bmatrix}}_{\vec{v}_2}, \underbrace{\begin{bmatrix}1\\-1\\1\end{bmatrix}}_{\vec{v}_3} \right\}$$

because $\vec{l} \perp P$, and $\begin{bmatrix}0\\1\\1\end{bmatrix} \neq k\begin{bmatrix}1\\1\\0\end{bmatrix}$ for $k \neq 0$

2 vectors from P, $\vec{l}$, all linearly independent

$\mathfrak{B}$-matrix: $A\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\1\\1\end{bmatrix} = \vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 \to \begin{bmatrix}1\\0\\0\end{bmatrix}$

$A\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix} = \vec{v}_2 + 0\vec{v}_1 + 0\vec{v}_3 \to \begin{bmatrix}0\\1\\0\end{bmatrix}$

$A\begin{bmatrix}1\\-1\\1\end{bmatrix} = \begin{bmatrix}3\\-3\\3\end{bmatrix} = 3\vec{v}_3 + 0\vec{v}_1 + 0\vec{v}_2 \to \begin{bmatrix}0\\0\\3\end{bmatrix}$

$\mathfrak{B}$-matrix: $\boxed{\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 3\end{bmatrix}}$ no

Use this page for any additional work. Make sure to clearly state which problem you are solving on this page.

Continuation of Problem 3, Part 1a)

2) closed under scalar multiplication ✓

$\vec{v} \in V$, prove $k\vec{v} \in V$ where $k \in \mathbb{R}$

$\vec{v} \cdot \vec{x} = \vec{v} \cdot \vec{y} \Rightarrow v_1 x_1 + v_2 x_2 + \dots v_n x_n = v_1 y_1 + v_2 y_2 + \dots v_n y_n$

Multiply both sides by k

$kv_1 x_1 + kv_2 x_2 + \dots kv_n x_n = kv_1 y_1 + kv_2 y_2 + \dots kv_n y_n$

$\rightarrow (k\vec{v}) \cdot \vec{x} = (k\vec{v}) \cdot \vec{y} \rightarrow k\vec{v} \in V$

V: closed under addition, closed under scalar multiplication

$\rightarrow$ V is a subspace $\vec{0} \in V$?

Use this page for any additional work. Make sure to clearly state which problem you are solving on this page.