Math 33A - Lectures 3 and 4
Fall 2018

Midterm 1

Instructions: You have 60 minutes to complete this exam. There are five questions, worth a total of 50 points. This test is closed book and closed notes. No calculator is allowed.

For full credit show all of your work legibly. Unless instructed otherwise, you need to justify your answers. Please write your solutions in the space below the questions; INDICATE if you go over the page and/or use the scrap pages at the end of this booklet.

Please take a moment to ensure that your booklet consists of ten pages, the last three being reserved for additional work.

Do not forget to write your full name, section and UID in the space below. For identification purposes, please sign below.

Full Name: ________
Student ID: ________
Lecture: 4
Section: 4E

Signature:

Question	Points	Score
1	8	8
2	10	10
3	11	8
4	10	10
5	11	9
Total:	50	45

Problem 1.

For each of the following sentences, give an example of a matrix A with the following properties, or explain why it is impossible.

(a) [4pts.] A is a 3×6 matrix with rank and nullity both equal to 3.

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

There are 3 pivots, so rank = 3
There are 3 free variables, so nullity = 3.
and A is a 3x6 matrix

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(b) [4pts.] A is a 6×3 matrix with rank and nullity both equal to 3.

This is not possible. According to the rank-nullity theorem, rank + nullity = m, where m is the number of columns in a matrix A. $3+3 \neq 3$, so no such matrix exists.

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Problem 2.

Recall that two vectors $\vec{v}, \vec{w} \in \mathbf{R}^n$ are perpendicular if their dot product is zero: $\vec{v} \cdot \vec{w} = \vec{0}$.

(a) [5pts.] Find a nonzero matrix A such that $A\vec{x}$ is perpendicular to the vector $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ for every $\vec{x} \in \mathbf{R}^3$. [You do not need to justify how you found A, but you do need to show that your choice of A satisfies the prescribed condition.]

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \begin{bmatrix} ax + by + cz \\ dx + ey + fz \\ gx + hy + iz \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0 \quad \text{so}$$

$ax + by + cz + dx + ey + fz + gx + hy + iz = 0$

$(a+d+g)x + (b+e+h)y + (c+f+i)z = 0$ for any x, y, z

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so basically, $a+d+g=0$ where $a,d,g,b,e,h,c,f,i \in \mathbb{R}$
$b+e+h=0$
$c+f+i=0$

so one such $A = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$

(b) [5pts.] For the matrix A you found in part (a), let $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ be the corresponding linear transformation. Find, with justification, a basis of the image of T.

A basis of the image of $T = \left\{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right\}$ since A only has one linearly independent column

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = -1\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

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Problem 3.

Let $\vec{x}, \vec{y}$ be two nonzero vectors in $\mathbf{R}^n$. Consider the set $(\vec{v}+\vec{w})\cdot\vec{n}$

$$V = \{\vec{v} \in \mathbf{R}^n \text{ such that } \vec{v}\cdot\vec{x} = \vec{v}\cdot\vec{y}\}.$$

$\vec{v}\cdot\vec{x} \cdot \vec{v}\cdot\vec{y}$

(a) [3pts.] Prove that V is a subspace of $\mathbf{R}^n$.

Condition 1

$\vec{v}=\vec{0}$ satisfies $\vec{v}\cdot\vec{x}=\vec{v}\cdot\vec{y}$. Therefore $\vec{0}\in V$, so Condition 1 holds ✓

Condition 2

The dot product is distributive, so if $\vec{v},\vec{w}\in V$, then

$(\vec{v}+\vec{w})\cdot\vec{x} = \vec{v}\cdot\vec{x}+\vec{w}\cdot\vec{x}$ and $(\vec{v}+\vec{w})\cdot\vec{y} = \vec{v}\cdot\vec{y}+\vec{w}\cdot\vec{y}$ ✓

Therefore, $(\vec{v}+\vec{w})\cdot\vec{x} = (\vec{v}+\vec{w})\cdot\vec{y}$, so Condition 2 holds

Condition 3

If $k\in\mathbb{R}$ and $\vec{v}\in V$, then $(k\vec{v})\cdot\vec{x} = k(\vec{v}\cdot\vec{x})$ and $(k\vec{v})\cdot\vec{y} = k(\vec{v}\cdot\vec{y})$

3/3 so $(k\vec{v})\cdot\vec{x} = (k\vec{v})\cdot\vec{y}$, so Condition 3 holds. ✓

Since all conditions are satisfied, V is a subspace of $\mathbb{R}^n$

(b) [3pts.] Let now $\vec{x} = \begin{bmatrix}2\\1\\0\\1\end{bmatrix}$ and $\vec{y} = \begin{bmatrix}1\\2\\-1\\0\end{bmatrix}$ be vectors in $\mathbf{R}^4$ and let V be defined as above.

Show that $\vec{v}_1 = \begin{bmatrix}1\\1\\-1\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix}1\\-1\\-1\\-1\end{bmatrix}$ belong to V.

$\vec{v}_1$

$\vec{v}_1\cdot\vec{x}$ $\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}\cdot\begin{bmatrix}2\\1\\0\\1\end{bmatrix} = 2+1+1 = 4$

$\vec{v}_1\cdot\vec{y}$ $\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\2\\-1\\0\end{bmatrix} = 1+2+1 = 4$

Since $\vec{v}_1\cdot\vec{x} = \vec{v}_1\cdot\vec{y}$, $\vec{v}_1\in V$

$\vec{v}_2$

$\vec{v}_2\cdot\vec{x} = \begin{bmatrix}0\\1\\1\\0\end{bmatrix}\cdot\begin{bmatrix}2\\1\\0\\1\end{bmatrix} = 1$

$\vec{v}_2\cdot\vec{y} = \begin{bmatrix}0\\1\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\2\\-1\\0\end{bmatrix} = 2-1 = 1$

Since $\vec{v}_2\cdot\vec{x} = \vec{v}_2\cdot\vec{y}$, $\vec{v}_2\in V$

3/3

$\vec{v}_3$

$\vec{v}_3\cdot\vec{x}$ $\begin{bmatrix}1\\-1\\-1\\-1\end{bmatrix}\cdot\begin{bmatrix}2\\1\\0\\1\end{bmatrix} = 2-1-1 = 0$

$\vec{v}_3\cdot\vec{y}$ $\begin{bmatrix}1\\-1\\-1\\-1\end{bmatrix}\cdot\begin{bmatrix}1\\2\\-1\\0\end{bmatrix} = 1-2+1 = 0$

Since $\vec{v}_3\cdot\vec{x} = \vec{v}_3\cdot\vec{y}$, $\vec{v}_3\in V$

This problem continues from the previous page. Recall that $\vec{x}, \vec{y}, \vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ are defined in question (b), and $V = \{\vec{v} \in \mathbf{R}^4 \text{ such that } \vec{v} \cdot \vec{x} = \vec{v} \cdot \vec{y}\}$.

(c) [5pts.] Prove that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis of V.

$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$ are linearly independent since to have v_3 be a linear combination of v_1 and v_2, v_1 needs to be multiplied by ~~a factor of~~ -1, which leaves -1 in the first row. However, v_2 has a 0 in the first row, so there's no possible combination of v_1 and v_2 to get v_3. That's the first condition of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ being a basis.

They also need to span all of V +1

This only checks that $\vec{v}_3$ is Not redundant, how about $\vec{v}_2$?

−1

2/5

Problem 4.

(a) [5pts.] Using row-reduction find, if it exists, the inverse of the matrix

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 7 & 1 & 0 & 0 \\ 2 & 5 & 8 & 0 & 1 & 0 \\ 3 & 6 & 9 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2-2R_1} \left[\begin{array}{ccc|ccc} 1 & 4 & 7 & 1 & 0 & 0 \\ 0 & -3 & -6 & -2 & 1 & 0 \\ 0 & -6 & -12 & -3 & 0 & 1 \end{array}\right] \xrightarrow{R_2/(-3)} \left[\begin{array}{ccc|ccc} 1 & 4 & 7 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2/3 & -1/3 & 0 \\ 0 & -6 & -12 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3+6R_2]{R_1-4R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -5/3 & 4/3 & 0 \\ 0 & 1 & 2 & 2/3 & -1/3 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right]$$

The rref(A) $\neq I_n$, so the inverse of A does not exist.

✓

(b) [5pts.] Let A be the matrix defined in (a). Find all solutions $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ of the system

$$A\vec{x} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$ [There could be none.]

$$\left[\begin{array}{ccc|c} 1 & 4 & 7 & -1 \\ 2 & 5 & 8 & 0 \\ 3 & 6 & 9 & 1 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 4 & 7 & -1 \\ 0 & -3 & -6 & 2 \\ 0 & -6 & -12 & 4 \end{array}\right] \xrightarrow{R_2/(-3)} \left[\begin{array}{ccc|c} 1 & 4 & 7 & -1 \\ 0 & 1 & 2 & -2/3 \\ 0 & -6 & -12 & 4 \end{array}\right]$$

$$\xrightarrow[R_3+6R_2]{R_1-4R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 5/3 \\ 0 & 1 & 2 & -2/3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so $x - z = 5/3$
$y + 2z = -2/3$
$z = s$ where $s \in \mathbb{R}$

$\rightarrow$ $x = s + 5/3$
$y = -2s - 2/3$
$z = s$

so all solutions of the system $A\vec{x} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are of the form

$\vec{x} = \begin{bmatrix} s+5/3 \\ -2s-2/3 \\ s \end{bmatrix}$ where $s \in \mathbb{R}$.

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Problem 5.

Let P be the plane in $\mathbf{R}^3$ given by the equation $x - y + z = 0$.

(a) [5pts.] Let $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ be the reflection across the plane P. Find the matrix of T with respect to the standard basis of $\mathbf{R}^3$.

$\text{refl}_P(\vec{v}) = 2\,\text{Proj}_L(\vec{v}) - \vec{v}$ ✓ where L is a line parallel to P that

so let $L = \text{span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$ – no, use $\begin{pmatrix}1\\-1\\1\end{pmatrix}$

let matrix of T be A such that $A = \begin{bmatrix} \text{refl}_P\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) & \text{refl}_P\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) & \text{refl}_P\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)\end{bmatrix}$

3/5

$$\text{refl}_P\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = 2\left(\left(\frac{\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\0\end{bmatrix}}{\begin{bmatrix}1\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\0\end{bmatrix}}\right)\begin{bmatrix}1\\1\\0\end{bmatrix}\right) - \begin{bmatrix}1\\0\\0\end{bmatrix} = \frac{2}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} - \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$\text{refl}_P\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = 2\left(\left(\frac{\begin{bmatrix}0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\0\end{bmatrix}}{\begin{bmatrix}1\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\0\end{bmatrix}}\right)\begin{bmatrix}1\\1\\0\end{bmatrix}\right) - \begin{bmatrix}0\\1\\0\end{bmatrix} = \frac{2}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} - \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$\text{refl}_P\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = 2\left(\left(\frac{\begin{bmatrix}0\\0\\1\end{bmatrix}\cdot\begin{bmatrix}-1\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\0\end{bmatrix}}\right)\begin{bmatrix}1\\1\\0\end{bmatrix}\right) - \begin{bmatrix}0\\0\\1\end{bmatrix} = \frac{2}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} - \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\-1\end{bmatrix}.$$

so $A = \begin{bmatrix}0&1&1\\1&0&1\\0&0&-1\end{bmatrix}$

(b) [6pts.] Find a basis $\mathfrak{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of $\mathbf{R}^3$ such that the $\mathfrak{B}$-matrix of T is diagonal. Write down the $\mathfrak{B}$-matrix of T.

Let $\vec{v}_1 = \begin{bmatrix}1\\-1\\1\end{bmatrix}$. Since $\vec{v}_1$ is orthogonal to the plane P,

$T(\vec{v}_1) = \begin{bmatrix}-1\\1\\-1\end{bmatrix}$

so $[T(\vec{v}_1)]_{\mathcal{B}} = \begin{bmatrix}-1\\0\\0\end{bmatrix}_{\mathcal{B}}$

let $\vec{v}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}$. Since $\vec{v}_2$ is on plane P,

$T(\vec{v}_2) = \begin{bmatrix}1\\1\\0\end{bmatrix}$

so $[T(\vec{v}_1)]_{\mathcal{B}} = \begin{bmatrix}0\\1\\0\end{bmatrix}_{\mathcal{B}}$

Let $\vec{v}_3 = \begin{bmatrix}0\\1\\1\end{bmatrix}$, which is also on plane P but not parallel to $\vec{v}_2$

so $T(\vec{v}_3) = \begin{bmatrix}0\\1\\1\end{bmatrix}$

and $[T(\vec{v}_3)]_{\mathcal{B}} = \begin{bmatrix}0\\0\\1\end{bmatrix}_{\mathcal{B}}$

So, a basis $\mathcal{B} = \left\{\begin{bmatrix}1\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$.

and the $\mathcal{B}$-matrix of $T = \begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}$