Problem 1.

For each of the following sentences, give an example of a matrix A with the following properties, or explain why it is impossible.

7. (a) [4pts.] A is a 3×6 matrix with rank and nullity both equal to 3.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

4

(b) [4pts.] A is a 6×3 matrix with rank and nullity both equal to 3.

Given and $n \times m$ matrix A, the rank-nullity theorem states that rank(A) + nullity (A) = m.

4

Substituting in the givens, we get $3+3=3$, which is not true; therefore, there is no such matrix

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Problem 2.

Recall that two vectors $\vec{v}, \vec{w} \in \mathbf{R}^n$ are perpendicular if their dot product is zero: $\vec{v} \cdot \vec{w} = \vec{0}$.

(a) [5pts.] Find a nonzero matrix A such that $A\vec{x}$ is perpendicular to the vector $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ for every $\vec{x} \in \mathbf{R}^3$. [You do not need to justify how you found A, but you do need to show that your choice of A satisfies the prescribed condition.]

5

$\vec{l} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \perp \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad \vec{l} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

A is the trans. matrix of $\text{proj}_L(\vec{x}) = \left[\text{proj}_L \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \; \text{proj}_L \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \; \text{proj}_L \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right]$

$\text{proj}_L \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \dfrac{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix}$ $\quad \text{proj}_L \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \vec{0}$ $\quad \text{proj}_L \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix}$

$A = \begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix}$

$A\vec{x} = \begin{bmatrix} x_1/2 - x_3/2 \\ 0 \\ -x_1/2 + x_3/2 \end{bmatrix}$ $\quad \begin{bmatrix} x_1/2 - x_3/2 \\ 0 \\ -x_1/2 + x_3/2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$

therefore all $A\vec{x}$ are perpendiculer to $\vec{v}$

(b) [5pts.] For the matrix A you found in part (a), let $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ be the corresponding linear transformation. Find, with justification, a basis of the image of T.

basis of im$(T) = \left\{ \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix} \right\}$ because the basis of the image of a transformation T is the span of the column space of its transformation matrix A (found above). However, the zero vector column is redundant as it can be written as 0(column 1) and column 3 is redundant as it can be written as $-$(column 1) $-$ since both columns are redundant (can be written as a linear combination of the last vector c_1/ are linearly dependent), they can be removed from the basis.

5

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(2 1) (2 3) 7

(4 2) (2 3) 14

Problem 3.

Let $\vec{x}, \vec{y}$ be two nonzero vectors in $\mathbf{R}^n$. Consider the set

$$V = \{\vec{v} \in \mathbf{R}^n \text{ such that } \vec{v} \cdot \vec{x} = \vec{v} \cdot \vec{y}\}.$$

(a) [3pts.] Prove that V is a subspace of $\mathbf{R}^n$.

① $\vec{0} \in V$ because $\vec{0} \cdot \vec{x} = \vec{0} \cdot y = 0$, so $\vec{v}$ can equal zero

② any $\vec{v} \in V$ can be written as $\vec{v}_1 + \vec{v}_2 \mid \vec{v}_1, \vec{v}_2 \in V$ because given $\vec{v} = \vec{v}_1 + \vec{v}_2$, $(\vec{v}_1 + \vec{v}_2) \cdot \vec{x} = (\vec{v}_1 + \vec{v}_2) \cdot \vec{y}$ must hold true

$$\vec{v}_1 \cdot \vec{x} + \vec{v}_2 \cdot \vec{x} = \vec{v}_1 \cdot \vec{y} + \vec{v}_2 \cdot \vec{y}$$

because $\vec{v}_1, \vec{v}_2 \in V$, $\vec{v}_1 \cdot \vec{x} = \vec{v}_1 \cdot \vec{y}$ and $\vec{v} \cdot \vec{x} = \vec{v}_2 \cdot \vec{y}$

Therefore, it holds true

③ given any $\vec{v} \in V$, $k\vec{v} \in V$ because $(k\vec{v}) \cdot \vec{x}$ can be rewritten as $k(\vec{v} \cdot \vec{x})$

so $(k\vec{v} \cdot \vec{x}) = (k\vec{v} \cdot \vec{y})$ holds true because it is equivalent to

$k(\vec{v} \cdot \vec{x}) = k(\vec{v} \cdot \vec{y})$

$\vec{v} \cdot \vec{x} = \vec{v} \cdot \vec{y}$, which is known to be true

3/3

(b) [3pts.] Let now $\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$ be vectors in $\mathbf{R}^4$ and let V be defined as above.

Show that $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$ belong to V.

3/3

V_1: $\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$

$4 = 4$

$\therefore V_1 \in V$

V_2: $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$

$1 = 1$

$\therefore V_2 \in V$

V_3: $\begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$

$0 = 0$

$\therefore V_3 \in V$

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This problem continues from the previous page. Recall that $\vec{x}, \vec{y}, \vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ are defined in question (b), and $V = \{\vec{v} \in \mathbf{R}^4 \text{ such that } \vec{v}\cdot\vec{x} = \vec{v}\cdot\vec{y}\}$.

(c) [5pts.] Prove that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis of V.

① v_1, v_2, and v_3 are linearly independent

$$\text{rref}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$ ✓

each column has a pivot so no columns are linearly dependent

② $\dim(V) = 3$

$\dim(V) \geq 3$ because at least 3 linearly independent vectors span it

$\dim(V) \leq 4$ because it exists with $\mathbb{R}^4$

$\dim(V) \neq 4$ because if it did, it would span all of $\mathbb{R}^4$ and it does not because some vectors $\vec{v} \in \mathbb{R}^4$ are not included in V

For example, $\begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$ b/c $\begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = 8$

$\neq$ ✓

$\begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} = 6$

Because basis $= \{v_1, v_2, v_3\}$ has 3 independent vectors $= \dim(V)$, $\{v_1, v_2, v_3\}$ fully spans V and is a basis of V

5/5

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Problem 4.

(a) [5pts.] Using row-reduction find, if it exists, the inverse of the matrix

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 7 & 1 & 0 & 0 \\ 2 & 5 & 8 & 0 & 1 & 0 \\ 3 & 6 & 9 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 4 & 7 & 1 & 0 & 0 \\ 0 & -3 & -6 & -2 & 1 & 0 \\ 0 & -6 & -12 & -3 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 7 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & -6 & -12 & -3 & 0 & 1 \end{array}\right] \to \begin{bmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \ldots$$

Does not exist because rank$(A) = 2 \to A$ is not invertible

(b) [5pts.] Let A be the matrix defined in (a). Find all solutions $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ of the system

$A\vec{x} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. [There could be none.]

$$\left[\begin{array}{ccc|c} 1 & 4 & 7 & -1 \\ 2 & 5 & 8 & 0 \\ 3 & 6 & 9 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 4 & 7 & -1 \\ 0 & -3 & -6 & 2 \\ 0 & -6 & -12 & 4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 7 & -1 \\ 0 & 1 & 2 & 2 \\ 0 & -6 & -12 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 0 & -1 & \\ 0 & 1 & 2 & \\ 0 & 0 & 0 & \cancel{16}\ 0 \end{array}\right]$$

$0 \neq 16 \to$ inconsistent system $\to$ No Solution

2

7

Problem 5.

Let P be the plane in $\mathbf{R}^3$ given by the equation $x - y + z = 0$.

(a) [5pts.] Let $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ be the reflection across the plane P. Find the matrix of T with respect to the standard basis of $\mathbf{R}^3$.

$\ell = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ $\quad proj_P = \vec{v} - proj_L \quad ref_P = 2proj_P - \vec{v} = 2\vec{v} - 2proj_L - \vec{v}$

$ref_P = \vec{v} - 2proj_L$

$\vec{v} - 2\frac{\vec{v}\cdot\ell}{\ell\cdot\ell}\ell$

A = matrix of T

$$ref_P\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 2\frac{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{pmatrix} 1/3 \\ 2/3 \\ -2/3 \end{pmatrix}$$

$$ref_P\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 2\frac{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$$

$$ref_P\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 2\frac{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$$

(b) [6pts.] Find a basis $\mathfrak{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of $\mathbf{R}^3$ such that the $\mathfrak{B}$-matrix of T is diagonal. Write down the $\mathfrak{B}$-matrix of T.

$v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

3/6

$\mathfrak{B} = \left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}$

$\mathfrak{B}$ matrix of $T = \left[T\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, T\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, T\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right]$

no.

$$\mathfrak{B}\text{-matrix of } T = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$$

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