Problem 1.

For each of the following sentences, give an example of a matrix A with the following properties, or explain why it is impossible.

(a) [4pts.] A is a 3×6 matrix with rank and nullity both equal to 3.

$$\boxed{A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}}$$ ✓

4

(b) [4pts.] A is a 6×3 matrix with rank and nullity both equal to 3.

Impossible $\rightarrow$ rank(A) + nullibility(A) = # of columns

$3 + 3 \neq 3$ ✗

4

$-\frac{1}{3} - \frac{1}{2} = \frac{-2}{6} \frac{-3}{6} = \frac{-5}{6}$, $\frac{1}{6}$, $\frac{1}{6} + \frac{1}{2} = \frac{1}{6} + \frac{3}{6} = \frac{4}{6} = \frac{2}{3}$

$1+1=2$

Problem 2.

Recall that two vectors $\vec{v}, \vec{w} \in \mathbf{R}^n$ are perpendicular if their dot product is zero: $\vec{v} \cdot \vec{w} = \vec{0}$.

$4+1+1=6$

(a) [5pts.] Find a nonzero matrix A such that $A\vec{x}$ is perpendicular to the vector $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ for every $\vec{x} \in \mathbf{R}^3$. [You do not need to justify how you found A, but you do need to show that your choice of A satisfies the prescribed condition.]

A is matrix of projection onto $x+y+z=0$?

Let $\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$\vec{\mu}_1 = \frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \vec{\mu}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$\text{proj}_{sp}(\vec{v}) = \text{proj}_{\vec{v}_1} + \text{proj}_{\vec{v}_2} = (\vec{\mu}_1 \cdot \vec{v})\vec{\mu}_1 + (\vec{\mu}_2 \cdot \vec{v})\vec{\mu}_2$

$\frac{1}{6}(2x-y-z)\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} + \frac{1}{2}(x-z)\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

5

$\text{proj}_{\vec{v}}(\vec{v}) = \begin{bmatrix} \frac{1}{3}(2x-y-z) + \frac{1}{2}(x-z) \\ -\frac{1}{6}(2x-y-z) \\ -\frac{1}{6}(2x-y-z) - \frac{1}{2}(x-z) \end{bmatrix}$

$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 7/6 \\ -1/3 \\ -5/6 \end{bmatrix}, T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1/3 \\ 1/6 \\ 1/6 \end{bmatrix}, T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -5/6 \\ 1/6 \\ 2/3 \end{bmatrix} \rightarrow \boxed{A = \begin{bmatrix} 7/6 & -1/3 & -5/6 \\ -1/3 & 1/6 & 1/6 \\ -5/6 & 1/6 & 2/3 \end{bmatrix}}$

(b) [5pts.] For the matrix A you found in part (a), let $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ be the corresponding linear transformation. Find, with justification, a basis of the image of T.

$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$

3

~~For all answers to be perpendicular to $\vec{v}$, T must be the orthogonal projection onto the plane perpendicular to $\vec{v}$.~~

The vectors $\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ are two linear independent vectors on that plane, and so they are a basis for the image of T.

Problem 3.

Let $\vec{x}, \vec{y}$ be two nonzero vectors in $\mathbf{R}^n$. Consider the set

$$V = \{\vec{v} \in \mathbf{R}^n \text{ such that } \vec{v} \cdot \vec{x} = \vec{v} \cdot \vec{y}\}.$$

(a) [3pts.] Prove that V is a subspace of $\mathbf{R}^n$.

1) $\vec{0} \in V \rightarrow \vec{0} \cdot \vec{x} = 0 \overset{?}{=} \vec{0} \cdot \vec{y} = 0$ ✓

2) $k\vec{v} \in V \rightarrow \frac{k\vec{v} \cdot \vec{x}}{k} = \frac{k\vec{v} \cdot y}{k} \Rightarrow \vec{v} \cdot \vec{x} = \vec{v} \cdot \vec{y}$ ✓ ✓

3) for $\vec{v}_1, \vec{v}_2 \in V$, $\vec{v}_1 + \vec{v}_2 \in V$

$(\vec{v}_1 + \vec{v}_2) \cdot \vec{x} = (\vec{v}_1 + \vec{v}_2) \cdot \vec{y}$

$\vec{v}_1 \cdot \vec{x} + \vec{v}_2 \cdot \vec{x} = \vec{v}_1 \cdot \vec{y} + \vec{v}_2 \cdot \vec{y}$ ✓

3/3

(b) [3pts.] Let now $\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$ be vectors in $\mathbf{R}^4$ and let V be defined as above.

Show that $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$ belong to V.

$\vec{v}_1$: $\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = 2+1+1=4 \overset{?}{=} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} = 1+2+1=4$ ✓

3/3

$\vec{v}_2$: $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = 0+1+0+0=1 \overset{?}{=} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} = 0+2-1+0=1$ ✓

$\vec{v}_3$: $\begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = 2-1+0-1=0 \overset{?}{=} \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} = 1-2+1+0=0$ ✓

This problem continues from the previous page. Recall that $\vec{x}, \vec{y}, \vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ are defined in question (b), and $V = \{\vec{v} \in \mathbf{R}^4 \text{ such that } \vec{v} \cdot \vec{x} = \vec{v} \cdot \vec{y}\}$.

(c) [5pts.] Prove that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis of V.

To prove that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis of V, we need to show that they're nonzero, in V, and linearly independent.

Check that they span V!!!

Nonzero: they're given in (b) as nonzero.

In V: Shown in part (b).

Linearly independent: Assume $\vec{v}_2$ is linearly dependent on $\vec{v}_1$. Then $\vec{v}_2 = a\vec{v}_1$.

$$\begin{bmatrix}0\\1\\1\\0\end{bmatrix} = a\begin{bmatrix}1\\1\\-1\\1\end{bmatrix} \rightarrow \begin{aligned} 0&=a\\ 1&=a\\ 1&=-a\\ 0&=a \end{aligned}$$

Clearly, a cannot equal 0, 1, and -1 at the same time. We have a contradiction, and $\vec{v}_2 \neq a\vec{v}_1$. ✓

$0 = a$
$1 \neq 0$ X

Assume $\vec{v}_3$ is linearly dependent on $\vec{v}_1$ and $\vec{v}_2$. Then $\vec{v}_3 = a\vec{v}_1 + b\vec{v}_2$.

$$\begin{bmatrix}1\\-1\\-1\\-1\end{bmatrix} = a\begin{bmatrix}1\\1\\-1\\1\end{bmatrix} + b\begin{bmatrix}0\\1\\1\\0\end{bmatrix} \rightarrow \begin{aligned} 1&=a\\ -1&=a+b\\ -1&=-a+b\\ -1&=a \end{aligned}$$

$1 \neq -1$

Clearly, this system has no solution, and we have a contradiction. $\vec{v}_3$ cannot be a linear combination of $\vec{v}_1$ and $\vec{v}_2$, so all vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ must be linearly independent.

Because $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are nonzero, in V, and linearly independent, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis of V. ✓

2/5

Problem 4.

(a) [5pts.] Using row-reduction find, if it exists, the inverse of the matrix

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The rank of the matrix A is not equal to 3, so the inverse of A does NOT exist. X

(b) [5pts.] Let A be the matrix defined in (a). Find all solutions $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ of the system

$$A\vec{x} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$ [There could be none.]

$$\left[\begin{array}{ccc|c} 1 & 4 & 7 & -1 \\ 2 & 5 & 8 & 0 \\ 3 & 6 & 9 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 4 & 7 & -1 \\ 0 & -3 & -6 & 2 \\ 0 & -6 & -12 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 4 & 7 & -1 \\ 0 & 1 & 2 & -2/3 \\ 0 & 1 & 2 & -2/3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 5/3 \\ 0 & 1 & 2 & -2/3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = 5/3 + x_3$

$x_2 = -2/3 - 2x_3$

$$\boxed{\vec{x} = \begin{bmatrix} 5/3 + x_3 \\ -2/3 - 2x_3 \\ x_3 \end{bmatrix}}$$

$\forall x_3 \in \mathbb{R}$

10

Problem 5.

Let P be the plane in $\mathbf{R}^3$ given by the equation $x - y + z = 0$.

(a) [5pts.] Let $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ be the reflection across the plane P. Find the matrix of T with respect to the standard basis of $\mathbf{R}^3$.

$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ ← normal

$\left.\begin{array}{l} \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ \vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \end{array}\right\}$ on plane

$T(\vec{v}_1) = -\vec{v}_1$

$T(\vec{v}_2) = \vec{v}_2$

$T(\vec{v}_3) = \vec{v}_3$

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 1$

(b) [6pts.] Find a basis $\mathfrak{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of $\mathbf{R}^3$ such that the $\mathfrak{B}$-matrix of T is diagonal. Write down the $\mathfrak{B}$-matrix of T.

Let $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and K = constant.

$T(\vec{v}_1) = c_1\vec{v}_1$

$T(\vec{v}_2) = c_2\vec{v}_2$

$T(\vec{v}_3) = c_3\vec{v}_3$

$T(\vec{v}) = K\vec{v}$

?/11