P

MIDTERM 2 A

11/22/2017

Math33A
Nadja Hempel
nadja@math.ucla.edu

Name: Steven La

UID: [redacted]

section: 4F w/ Blaine Talbut (Thurs 3-3:50 PM)

Problem	Points	Score
1	10	10
2	8	8
3	12	12
SA	10	10
Total	40	40

Instructions

- No notes, personal aids or calculators are permitted.
- This exam consists of 9 pages. Take a moment to make sure you have all pages.
- You have 45 minutes.
- Answer all questions in the space provided. If you require more space to write your answer, you may continue on the extra page at the end of the exam (make it clear if you do).
- Explain your work! Little or no points will be given for a correct answer with no explanation of how you got it.
- In particular, you have to write down all row operations for full credit.

(1) (10pt) Let $\mathcal{B} = ((2,1),(1,1))$ and consider a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$.

(a) Show that $\mathcal{B}$ is a basis of $\mathbb{R}^2$. (3pt)

$B = \left(\begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right)$. Since $\nexists\ k, k \in \mathbb{R}$ st

$k\vec{0} = \begin{pmatrix}2\\1\end{pmatrix}$ and $k\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$,

the vectors in B are lin. ind.

3

Also, since $\dim(\mathbb{R}^2) = 2$, we have 2 lin. ind. vectors, which also span $\mathbb{R}^2$. This implies that B is indeed, a basis of $\mathbb{R}^2$.

(b) Calculate $[(1,0)]_{\mathcal{B}}$ and $[(0,1)]_{\mathcal{B}}$. (2pt)

$\left[\begin{pmatrix}1\\0\end{pmatrix}\right]_{\mathcal{B}} \Rightarrow \begin{pmatrix}1\\0\end{pmatrix} = (1)\begin{pmatrix}2\\1\end{pmatrix} + (-1)\begin{pmatrix}1\\1\end{pmatrix}$

✓ $\begin{pmatrix}2\\1\end{pmatrix} + \begin{pmatrix}-1\\-1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$ ✓

$$\boxed{\left[\begin{pmatrix}1\\0\end{pmatrix}\right]_{\mathcal{B}} = \begin{bmatrix}1\\-1\end{bmatrix}}$$

$\left[\begin{pmatrix}0\\1\end{pmatrix}\right]_{\mathcal{B}} \Rightarrow \begin{pmatrix}0\\1\end{pmatrix} = (-1)\begin{pmatrix}2\\1\end{pmatrix} + (2)\begin{pmatrix}1\\1\end{pmatrix}$

2

✓ $\begin{pmatrix}-2\\-1\end{pmatrix} + \begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$ ✓

$$\boxed{\left[\begin{pmatrix}0\\1\end{pmatrix}\right]_{\mathcal{B}} = \begin{bmatrix}-1\\2\end{bmatrix}}$$

Suppose that $[T]_{\mathcal{B}} = \overbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}^{B}$.

(c) Calculate $T(2,1)$ and $T(1,1)$ (in the standard basis). (2pt)

$$\left[T\begin{pmatrix}2\\1\end{pmatrix}\right]_\beta = B\left[\begin{pmatrix}2\\1\end{pmatrix}\right]_\beta \rightarrow \begin{pmatrix}2\\1\end{pmatrix} = 1\begin{pmatrix}2\\1\end{pmatrix} + 0\cancel{\begin{pmatrix}1\\1\end{pmatrix}} \Rightarrow \left[\begin{pmatrix}2\\1\end{pmatrix}\right]_\beta = \begin{pmatrix}1\\0\end{pmatrix}$$

$$\left[T\begin{pmatrix}2\\1\end{pmatrix}\right]_\beta = \begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a\\c\end{pmatrix}$$

2

$$\Rightarrow T\begin{pmatrix}2\\1\end{pmatrix} = a\begin{pmatrix}2\\1\end{pmatrix} + c\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2a\\a\end{pmatrix} + \begin{pmatrix}c\\c\end{pmatrix} = \boxed{\begin{pmatrix}2a+c\\a+c\end{pmatrix} = T\begin{pmatrix}2\\1\end{pmatrix}}$$

$$\left[T\begin{pmatrix}1\\1\end{pmatrix}\right]_\beta = B\left[\begin{pmatrix}1\\1\end{pmatrix}\right]_\beta \rightarrow \begin{pmatrix}1\\1\end{pmatrix} = 0\begin{pmatrix}2\\1\end{pmatrix} + 1\begin{pmatrix}1\\1\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}b\\d\end{pmatrix}$$

$$T\begin{pmatrix}1\\1\end{pmatrix} = b\begin{pmatrix}2\\1\end{pmatrix} + d\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2b\\b\end{pmatrix} + \begin{pmatrix}d\\d\end{pmatrix} = \boxed{\begin{pmatrix}2b+d\\b+d\end{pmatrix} = T\begin{pmatrix}1\\1\end{pmatrix}}$$

(d) What is the standard matrix of T? (3pt)

$$B = S^{-1}AS, \quad S = \begin{pmatrix}2&1\\1&1\end{pmatrix}$$

$$SBS^{-1} = A$$

$$\Rightarrow \underbrace{\begin{pmatrix}2&1\\1&1\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}}\begin{pmatrix}1&-1\\-1&2\end{pmatrix}$$

3

$$\begin{pmatrix}2a+c & 2b+d\\ a+c & b+d\end{pmatrix}\begin{pmatrix}1&-1\\-1&2\end{pmatrix} = A$$

$$\boxed{A = \begin{pmatrix}2a+c-2b-d & -2a-c+4b+2d\\ a+c-b-d & -a-c+2b+2d\end{pmatrix}}$$

S^{-1}:

$$\begin{pmatrix}2&1\\1&1\end{pmatrix} : \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\begin{pmatrix}1&1\\2&1\end{pmatrix} : \begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$$\begin{pmatrix}2&2\\0&-1\end{pmatrix} = \begin{pmatrix}0&2\\1&-2\end{pmatrix}$$

$$\begin{pmatrix}1&1\\0&-1\end{pmatrix} = \begin{pmatrix}0&1\\1&-2\end{pmatrix}$$

$$\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \begin{pmatrix}1&-1\\1&-2\end{pmatrix}$$

$$\begin{pmatrix}1&0\\0&1\end{pmatrix} = \underbrace{\begin{pmatrix}1&-1\\-1&2\end{pmatrix}}_{S^{-1}}$$

Every $\vec{y} \in \mathbb{R}^m$ is mapped onto.

(2) ~~(8pt) Consider~~ an onto (surjective) linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ and a basis $\vec{v_1}, \dots, \vec{v_n}$ of $\mathbb{R}^n$.

(a) ~~Does $T(\vec{v_1})$~~, $\dots, T(\vec{v_n})$ in $\mathbb{R}^m$ span $\mathbb{R}^m$? Justify your answer. (3pt)

B/C $\vec{v_1}, \dots, \vec{v_n}$ is a basis of $\mathbb{R}^n$, it must span all of $\mathbb{R}^n$. Therefore, if we apply T to this basis, $T(\vec{v_1}), \dots, T(\vec{v_n})$ must also span $\mathbb{R}^m$ since T is defined to be onto, which means that $\forall \vec{y} \in \mathbb{R}^m$, there is a corresponding $\vec{x} \in \mathbb{R}^n$ s.t. $T(\vec{x}) = \vec{y}$. ✓

(b) What is the relationship between n and m? Justify your answer (2pt)

Because $T(\vec{v_1}), \dots, T(\vec{v_n})$ spans $\mathbb{R}^m$, $\boxed{n \geq m}$ ✓ since onto transformation implies that the matrix A assoc. w/ T must be consistent. This only happens if we have free variables ≥ 0. (ie $n \geq m$) ✓

(c) Under which condition (on n and m) is $T(\vec{v_1}), \dots, T(\vec{v_n})$ a basis for $\mathbb{R}^m$? Justify your answer. (3pt)

$\boxed{n = m}$ is the only condition, since $T(\vec{v_1}), \dots, T(\vec{v_n})$ already span $\mathbb{R}^m$, but in order to be a basis, $T(\vec{v_1}), \dots, T(\vec{v_n})$ must be lin ind, too. A way to show this is if $n = \dim(\mathbb{R}^m) = m$ then the vectors $T(\vec{v_1}), \dots, T(\vec{v_n})$ must be a basis. (And thus, lin. ind.) ✓

$\begin{matrix}20\\16\end{matrix}$ -36 $\quad \frac{5}{3}+\frac{4}{3}=\frac{9}{3}$ $\quad 8+1=9$ $\quad \frac{2}{3}-\frac{2}{3}$

$\frac{1}{3}+\frac{8}{3}$

$12 \swarrow \vec{v}_1$

(3) (12pt) Perform the Gram-Schmidt process on ((1, 0, 2, 2), (5, 4, 2, 0), (1, −1, 4, 0)) and find the QR factorization.

$$M = \left(\underset{\vec{v}_1}{\begin{pmatrix}1\\0\\2\\2\end{pmatrix}} \underset{\vec{v}_2}{\begin{pmatrix}5\\4\\2\\0\end{pmatrix}} \underset{\underset{3\vec{u}_1}{\vec{v}_3}}{\begin{pmatrix}1\\-1\\4\\0\end{pmatrix}} \right)$$

$$\vec{u}_1 : \frac{1}{\sqrt{1+0+4+4}} \begin{pmatrix}1\\0\\2\\2\end{pmatrix} \to \frac{1}{3}\begin{pmatrix}1\\0\\2\\2\end{pmatrix} = \begin{pmatrix}1/3\\0\\2/3\\2/3\end{pmatrix} = \vec{u}_1$$

$$3\vec{u}_1 = \vec{v}_1$$

$$\vec{u}_1$$

$6\vec{u}_2$

$$\vec{u}_2 : \quad v_2^{\perp} = \vec{v}_2 - \operatorname{proj}_V(\vec{v}_2) \quad \vec{u}_1$$

$5u_1 + 6u_2 = \vec{v}_2$

$6u_2 = v_2^{\perp}$

$$\begin{pmatrix}5\\4\\2\\0\end{pmatrix} - \left[\underbrace{\left(\begin{pmatrix}5\\4\\2\\0\end{pmatrix} \cdot \begin{pmatrix}1/3\\0\\2/3\\2/3\end{pmatrix}\right)}_{3} \begin{pmatrix}1/3\\0\\2/3\\2/3\end{pmatrix} \right] \to \begin{pmatrix}5\\4\\2\\0\end{pmatrix} - \left[3\begin{pmatrix}1/3\\0\\2/3\\2/3\end{pmatrix} \right]$$

$$\begin{pmatrix}5\\4\\2\\0\end{pmatrix} - \begin{pmatrix}1\\0\\2\\2\end{pmatrix} = \underbrace{\begin{pmatrix}4\\4\\0\\-2\end{pmatrix}}_{v_2^{\perp}}$$

$$\vec{u}_2 = \frac{1}{\sqrt{16+16+4}} \overset{v_2^{\perp}}{\begin{pmatrix}4\\4\\0\\-2\end{pmatrix}} = \frac{1}{6}\begin{pmatrix}4\\4\\0\\-2\end{pmatrix} = \begin{pmatrix}2/3\\2/3\\0\\-1/3\end{pmatrix} = \vec{u}_2$$

$3\vec{u}_1 \quad v_2^{\perp}$

$$\vec{u}_3 : \quad v_3^{\perp} = \vec{v}_3 - \operatorname{proj}_{V_2}(\vec{v}_3)$$

$$\begin{pmatrix}1\\-1\\4\\0\end{pmatrix} - \left[\underbrace{\left(\begin{pmatrix}1\\-1\\4\\0\end{pmatrix} \cdot \begin{pmatrix}1/3\\0\\2/3\\2/3\end{pmatrix}\right)}_{3} \underset{\vec{u}_1}{\begin{pmatrix}1/3\\0\\2/3\\2/3\end{pmatrix}} + \underbrace{\left(\begin{pmatrix}1\\-1\\4\\0\end{pmatrix} \cdot \begin{pmatrix}2/3\\2/3\\0\\-1/3\end{pmatrix}\right)}_{0} \overset{\vec{u}_2}{\begin{pmatrix}2/3\\2/3\\0\\-1/3\end{pmatrix}} \right]$$

$$\begin{pmatrix}1\\-1\\4\\0\end{pmatrix} - \begin{pmatrix}1\\0\\2\\2\end{pmatrix} = \begin{pmatrix}0\\-1\\2\\-2\end{pmatrix} \longrightarrow \vec{u}_3 = \frac{1}{\sqrt{1+4+4}} \begin{pmatrix}0\\-1\\2\\-2\end{pmatrix} = \overset{\vec{u}_3}{\begin{pmatrix}0\\-1/3\\2/3\\-2/3\end{pmatrix}}$$

$$\boxed{\underset{M}{\begin{pmatrix}1 & 5 & 1\\0 & 4 & -1\\2 & 2 & 4\\2 & 0 & 0\end{pmatrix}} = \underset{Q}{\begin{pmatrix}1/3 & 2/3 & 0\\0 & 2/3 & -1/3\\2/3 & 0 & 2/3\\2/3 & -1/3 & -2/3\end{pmatrix}} \underset{R}{\begin{pmatrix}3 & 3 & 3\\0 & 6 & 0\\0 & 0 & 3\end{pmatrix}}}$$

$3\vec{u}_3 = \vec{v}_3^{\perp}$

$3\vec{u}_3 + 3\vec{u}_1$

$0\ 6\ -3 \mid -2 \mid 1$

$\mid 0 \mid 0$

$1 \mid 4 \mid -2$

$0-(-2)\ 1 \quad 0-(-2)$

1

$4-(-2)\ 5 \quad 0-1-$

-1

$2-1=-3$

✓ $\begin{pmatrix}1\\1\\1\end{pmatrix}$ ✓ $\begin{pmatrix}-2\\0\\4\end{pmatrix}$ $\begin{pmatrix}1\\0\\-2\end{pmatrix}$ ~~$\begin{pmatrix}-1\\1\\5\end{pmatrix}$~~

$3 \quad \begin{pmatrix}-2\\0\\4\end{pmatrix} \quad 7 \quad \left(-\tfrac{1}{2}\right)\begin{pmatrix}-2\\0\\4\end{pmatrix}$

3

3

$\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 6 & -3 \end{array} \quad \left\{ \begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 2 & -1 \\ 0 & 6 & -3 \end{array} \right.$

$4-(-2)\ 3$

$-2-1=$

I. SHORT ANSWERS

(10pt)

Instructions: The following problems have a short answer. No reason needs to be given.

$\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & \boxed{-1/2} \\ 0 & 0 & 0 \end{array}$

2 (1) ~~(2pt)~~ Which of the following set are subspaces (circle them)?

[✓ (a)] $\{(x, y, z) : x = y - 4z, y + 2z = 0\}$.

~~(b) $\{(x, y, z) : x = 2 + y + 2z\}$.~~ $\longrightarrow \begin{pmatrix}0\\0\\0\end{pmatrix}$ ✗

~~(c) $\{(x, y, z, t) : x^2 - z = 0\}$.~~

[✓ (d)] $\{(x, y, z, t) : x = y\}$

$\begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}2\\2\\2\end{pmatrix} =$

$2-2$

1

(2) (2pt) Identify the redundant vectors of $\{(1, 1, 1), (-2, 0, 4), (1, 0, -2), (-1, 1, 5)\}$.

2 Answer:

$$\boxed{\begin{pmatrix}1\\0\\-2\end{pmatrix}, \begin{pmatrix}-1\\1\\5\end{pmatrix}}$$

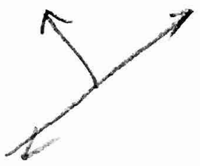
$\begin{vmatrix} & 0 & \\ 1 & 2 & 2 \\ -4 & 1 & 1 \end{vmatrix}$

$\langle 0, -9, 9\rangle$

$\begin{pmatrix}1\\0\\-2\end{pmatrix}$

$1-(-8)$

$1+8=9$



(3) (1pt) What is the dimension of span$((1, 1, 1), (-2, 0, 4), (1, 0, -2), (-1, 1, 5))$ (same vectors as above)

$1-(-8) \quad 1$

Answer:

$$\boxed{\text{dimension} = 2}$$

$\begin{pmatrix}1\\2\\2\end{pmatrix} \to \begin{pmatrix}-4\\1\\1\end{pmatrix}$

$T(\vec{v}_1) = c_1\vec{v}_1$

$T\begin{pmatrix}1\\2\\2\end{pmatrix} = 1\begin{pmatrix}1\\2\\2\end{pmatrix}$ 3

(4) (3pt) Find a basis $\mathcal{B}$ of $\mathbb{R}^3$ such that the $\mathcal{B}$-matrix of the reflection along the line spanned by $(1, 2, 2)$ is diagonal.

$-4+2+2=0$ ✓

$-4+2+2=0$ ✓

Answer:

$T\begin{pmatrix}-4\\1\\1\end{pmatrix} = -\begin{pmatrix}-4\\1\\1\end{pmatrix}$

$$\boxed{\mathcal{B} = \left(\begin{pmatrix}1\\2\\2\end{pmatrix}, \begin{pmatrix}-4\\1\\1\end{pmatrix}, \begin{pmatrix}0\\-9\\9\end{pmatrix}\right)}$$ ✓

$0+(-18)+18$

$\begin{matrix}1 & 0\\0 & -1\\0 & 0\end{matrix}$

2 (5) (2pt) Find the length of the vector $(6, -1, 1, -1, 3, 1)$

Answer:

$\|\vec{x}\| = \sqrt{\vec{x} \circ \vec{x}}$

$= \sqrt{36+1+1+1+9+1} = \sqrt{49} = \textcircled{7}$