MATH 33A Midterm 2 Feb. 25, 2019

STUDENT NAME: ______________________________

STUDENT ID NUMBER: ______________________________

DISCUSSION SECTION NUMBER: ______________________________

Directions

Answer each question in the space provided. Please write clearly and legibly. Show all of your work—your work must both justify and clearly identify your final answer. No books, notes or calculators are allowed. You must simplify results of function evaluations when it is possible to do so.

For instructor use only

Page	Points	Score
2	8	8
3	8	8
4	4	4
5	8	8
6	6	4
7	10	10
8	6	6
Total:	50	48

1. Consider the matrix

$$A = \begin{pmatrix} 1 & 5 & -3 \\ 2 & 0 & 4 \\ 2 & 2 & 2 \end{pmatrix}$$

4 (a) [4 pts] Find a basis for ker(A), which is one-dimensional.

Therefore, one of the basis in ker(A) is $\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ ✓

2 (b) [2 pts] Using your kernel computation, or otherwise, write down a linear relation between the columns of A that shows that the third column can be viewed as redundant. No further explanation necessary.

should explicitly write third column = ______

The relation is: $-2\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} = 0$

2 (c) [2 pts] Write down a basis for im(A). You may use any of the previous parts of the problem with no further explanation, even if you couldn't solve them. Or you may solve this problem from scratch using row reduction.

$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}$. ✓

(d) [8 pts] Convert your answer from the previous part of the problem into an orthonormal basis for im(A). (Keep your work well-organized!)

An orthonormal basis of is $\left\{ \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}, \begin{pmatrix} \frac{2\sqrt{5}}{5} \\ \frac{-\sqrt{5}}{5} \\ \text{[illegible]} \end{pmatrix} \right\}$

(e) [4 pts] Use your work from the previous part of the problem to write a $M = QR$ factorization for a relevant matrix M. (Part of the problem is deciding what M should be!)

$$\begin{bmatrix} 1 & 5 \\ 2 & 0 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2\sqrt{5}}{5} \\ \frac{2}{3} & \frac{-\sqrt{5}}{5} \\ \frac{2}{3} & 0 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & 2\sqrt{5} \end{bmatrix}$$

2. Suppose we have a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$, and suppose further that we have a subspace V in the range $\mathbb{R}^n$. We will define a new subset of the *domain* $\mathbb{R}^m$ called the *pre-image* of V as follows:

$$\text{PreIm}(V) := \{\vec{x} \text{ in } \mathbb{R}^m \mid T(\vec{x}) \text{ is in } V\}.$$

That is, PreIm(V) consists of all vectors in $\mathbb{R}^m$ that get mapped by T to a vector that is in V.

(a) [8 pts] Prove that PreIm(V) is a subspace of the domain $\mathbb{R}^m$ by checking all three necessary conditions.

To prove PreIm(V) is a subspace, we need to check the following 3 things.

1. $\vec{0}$ is in PreIm(V).

We have $\vec{0}$ in $\mathbb{R}^m$ that goes to $T(\vec{0}) = \vec{0}$ has a linear transformation, which ~~T~~ is in V since V is a subspace. Therefore, $\vec{0}$ is in PreIm(V).

2. PreIm(V) closed under addition.

If we have $\vec{x}_1, \vec{x}_2$ in PreIm(V)

then $T(\vec{x}_1), T(\vec{x}_2)$ is in V.

since V is a subspace and T is linear.

$T(\vec{x}_1 + \vec{x}_2) = T(\vec{x}_1) + T(\vec{x}_2)$ is also in V.

So $\vec{x}_1 + \vec{x}_2$ is in PreIm(V), PreIm(V) is closed under addition.

3. PreIm(V) closed under scalar multiplication.

if $\vec{x}_1$ is in PreIm(V)

Then $T(\vec{x}_1)$ is in V

Since V is a subspace, and T is linear.

$T(k\vec{x}_1) = k\,T(\vec{x}_1)$ is also in V.

So $k\vec{x}_1$ is in PreIm(V), PreIm(V) closed under scalar multiplication

So PreIm(V) is a subspace as all 3 condition met.

2 (b) [2 pts] If we had chosen the subspace V to be all of the range $\mathbb{R}^n$, what subspace would $\mathrm{PreIm}(V)$ be? No explanation necessary.

PreIm(V) will be all of R^m

✓

2 (c) [2 pts] If we had chosen the subspace V to be the zero subspace $\{\vec{0}\}$ in the range $\mathbb{R}^n$, what subspace would $\mathrm{PreIm}(V)$ be? No explanation necessary.

PreIm(V) will be ker(T).

✓

0 (d) [2 pts] Circle the one correct statement out of the four choices below (hint: think about $T : \mathrm{PreIm}(V) \to V$ rather than $T : \mathbb{R}^m \to \mathbb{R}^n$).

- $\dim(\mathrm{PreIm}(V)) = \dim(V)$ regardless of V and/or T.
- X $\dim(\mathrm{PreIm}(V)) \geq \dim(V)$ regardless of V and/or T. T does not need to surject onto V
- $\dim(\mathrm{PreIm}(V)) \leq \dim(V)$ regardless of V and/or T.
- In some cases $\dim(\mathrm{PreIm}(V)) > \dim(V)$. In other cases $\dim(\mathrm{PreIm}(V)) < \dim(V)$. Finally, there are also cases where $\dim(\mathrm{PreIm}(V)) = \dim(V)$.

3. Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ be a basis for $\mathbb{R}^2$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ be two linear transformations that treat this basis in the following manner:

$$T(\vec{v}_1) = \vec{v}_2, \quad T(\vec{v}_2) = \vec{v}_1, \qquad S(\vec{v}_1) = \vec{v}_1 - \vec{v}_2, \quad S(\vec{v}_2) = \vec{v}_2$$

4

(a) [4 pts] Write down the matrix B for the composition $T \circ S$ in $\mathcal{B}$-coordinates.

The matrix is

$$\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

1 0

6

(b) [6 pts] Prove that, if we assume that S w[illegible] an orthogonal transformation, then the basis vectors $\vec{v}_1$ and $\vec{v}_2$ *cannot* be orthogonal to each other.

if S is orthogonal, S preserves length, so

$$\|\vec{V}_1\| = \|\vec{V}_1 - \vec{V}_2\|$$

$$V_1 \cdot V_1 = (V_1 - V_2) \cdot (V_1 - V_2)$$

$$V_1 \cdot V_1 = V_1 \cdot V_1 - 2V_1 \cdot V_2 + V_2 \cdot V_2$$

$$0 = V_2 \cdot V_2 - 2V_1 \cdot V_2$$

if $\vec{V}_1$ and $\vec{V}_2$ are orthogonal, $V_1 \cdot V_2 = 0$.

Then $V_2 \cdot V_2 = 0$, or $\|V_2\| = 0$.

Then V_2 cannot be part of the basis, as $\vec{0}$ is always redundent.

So $\vec{V}_1$ and $\vec{V}_2$ ~~are not or~~ cannot be orthogonal to eachoth

4. Multiple choice and/or true and false (circle one answer only; no justification needed).
   In all of the problems below, A is an $\overset{3 \times 1}{nxm}$ matrix. $\mathbb{R}^m \to \mathbb{R}^n$

   (a) [2 pts] What can we say about $\dim(\text{im}(A)) + \dim(\ker(A))$?

   Always $= n$ Always $= m$ Neither of these

   $$\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

   (b) [2 pts] What can we say about $\dim(\ker(A))$?

   nullity

   Always $< n$ Always $> n$ Neither of these

   (c) [2 pts] If B is a pxn matrix, then we must have $\text{rank}(BA) \leq \text{rank}(A)$.

   TRUE FALSE

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