1. Consider the matrix

$$A = \begin{pmatrix} 1 & 5 & -3 \\ 2 & 0 & 4 \\ 2 & 2 & 2 \end{pmatrix} \begin{matrix} \\ -2\mathrm{I} \\ -2\mathrm{I} \end{matrix} \begin{bmatrix} 1 & 5 & -3 \\ 0 & -10 & 10 \\ 0 & -8 & 8 \end{bmatrix} \begin{matrix} \\ \div -10 \\ \div -8 \end{matrix} \rightarrow \begin{bmatrix} 1 & 5 & -3 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

4 (a) [4 pts] Find a basis for $\ker(A)$, which is one-dimensional.

$A\vec{x} = \vec{0}$ $\left[\begin{array}{ccc|c} 1 & 5 & -3 & 0 \\ 2 & 0 & 4 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right] \begin{matrix} \\ -2\mathrm{I} \\ -2\mathrm{I} \end{matrix} \left[\begin{array}{ccc|c} 1 & 5 & -3 & 0 \\ 0 & -10 & 10 & 0 \\ 0 & -8 & 8 & 0 \end{array}\right] \begin{matrix} \\ \div -10 \\ \div -8 \end{matrix} \rightarrow \left[\begin{array}{ccc|c} 1 & 5 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \begin{matrix} -5\mathrm{II} \\ \rightarrow \\ -\mathrm{II} \end{matrix}$

$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $x_3 = t$, $x_2 = t$, $x_1 = -2t$ $\quad \vec{x} = \begin{bmatrix} -2t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

free

$\boxed{\text{basis for } \ker(A) \text{ is } \left\{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right\}}$ ✓

$-2\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2+5 \\ -4+0 \\ -4+2 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix}$

2 (b) [2 pts] Using your kernel computation, or otherwise, write down a linear relation between the columns of A that shows that the third column can be viewed as redundant. No further explanation necessary.

1st col $= \vec{v}_1$ 2nd col $= \vec{v}_2$ 3rd col $= \vec{v}_3$

$0 = -2\vec{v}_1 + \vec{v}_2 + \vec{v}_3$

$\vec{v}_3 = 2\vec{v}_1 - \vec{v}_2$ ✓

2 (c) [2 pts] Write down a basis for $\mathrm{im}(A)$. You may use any of the previous parts of the problem with no further explanation, even if you couldn't solve them. Or you may solve this problem from scratch using row reduction.

first two columns have pivots,
so basis for $\mathrm{Im}(A)$ is $\left\{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}\right\}$ ✓

(d) [8 pts] Convert your answer from the previous part of the problem into an orthonormal basis for im(A). (Keep your work well-organized!).

$\mathcal{B}$ is basis of Im(A): $\mathcal{B} = \left\{ \underbrace{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}}_{\vec{v}_2} \right\}$

$\mathcal{U}$ is O-N basis of Im(A):

$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{\vec{v}_1}{\sqrt{1+4+4}} = \frac{\vec{v}_1}{3} = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

$\vec{u}_1 \cdot \vec{v}_2 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix} = \frac{5}{3} + \frac{4}{3} = \frac{9}{3} = 3$

$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 = \vec{v}_2 - 3\vec{u}_1 = \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix}$

$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{\vec{v}_2^{\perp}}{\sqrt{16+4}} = \frac{\vec{v}_2^{\perp}}{\sqrt{20}} = \frac{1}{2\sqrt{5}}\begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

$$\boxed{\mathcal{U} = \left\{ \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right\}}$$

(e) [4 pts] Use your work from the previous part of the problem to write a $M = QR$ factorization for a relevant matrix M. (Part of the problem is deciding what M should be!)

$$M = \begin{bmatrix} 1 & 5 \\ 2 & 0 \\ 2 & 2 \end{bmatrix} \qquad Q = \begin{bmatrix} 1/3 & 2/\sqrt{5} \\ 2/3 & -1/\sqrt{5} \\ 2/3 & 0 \end{bmatrix}$$

Columns are vectors from $\mathcal{B}$

columns are vectors from $\mathcal{U}$, O-N basis

$$R = \begin{bmatrix} \|\vec{v}_1\| & (\vec{u}_1 \cdot \vec{v}_2) \\ 0 & \|\vec{v}_2{}^{\perp}\| \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 0 & 2\sqrt{5} \end{bmatrix}$$

$$\boxed{\underset{M}{\begin{bmatrix} 1 & 5 \\ 2 & 0 \\ 2 & 2 \end{bmatrix}} = \underset{Q}{\begin{bmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} \\ \frac{2}{3} & \frac{-1}{\sqrt{5}} \\ \frac{2}{3} & 0 \end{bmatrix}} \underset{R}{\begin{bmatrix} 3 & 3 \\ 0 & 2\sqrt{5} \end{bmatrix}}}$$

✓

2. Suppose we have a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$, and suppose further that we have a subspace V in the range $\mathbb{R}^n$. We will define a new subset of the *domain* $\mathbb{R}^m$ called the *pre-image* of V as follows:

$$\text{PreIm}(V) := \{\vec{x} \text{ in } \mathbb{R}^m \,|\, T(\vec{x}) \text{ is in } V\}.$$

That is, PreIm(V) consists of all vectors in $\mathbb{R}^m$ that get mapped by T to a vector that is in V.

(a) [8 pts] Prove that PreIm(V) is a subspace of the domain $\mathbb{R}^m$ by checking all three necessary conditions.

1) $\vec{0}$ is in PreIm(V)

Since V is a subspace, V must contain $\vec{0}_n$.

Since T is a linear transformation, $T(\vec{0}_m)$ must $= \vec{0}_n$

$\therefore$ since $\vec{0}$ in $\mathbb{R}^m$ gets mapped to $\vec{0}$ in $\mathbb{R}^n$, which is also in V, PreIm(V) will also contain $\vec{0}$ in $\mathbb{R}^m$

2) closed under addition

if $\vec{u}, \vec{v}$ in PreIm(V), $T(\vec{u})$ and $T(\vec{v})$ are in V

Since T is linear transf., $T(\vec{u}+\vec{v}) = T(\vec{u})+T(\vec{v})$, and since V is a known subspace, $T(\vec{u})+T(\vec{v})$ is in V.

so therefore, since $T(\vec{u}+\vec{v})$ is in V, $(\vec{u}+\vec{v})$ must also be in PreIm(V) and PreIm(V) is closed under addition

3) closed under scalar multiplication

if $\vec{u}$ is in PreIm(V), $T(\vec{u})$ is in V, and since V is a subspace, $T(k\vec{u}) = kT(\vec{u})$ is also in V

therefore, $k\vec{u}$ is also in PreIm(V), and PreIm(V) is closed under scalar mult.

2 (b) [2 pts] If we had chosen the subspace V to be all of the range $\mathbb{R}^n$, what subspace would $\text{PreIm}(V)$ be? No explanation necessary.

$\mathbb{R}^m$ ✓

2 (c) [2 pts] If we had chosen the subspace V to be the zero subspace $\{\vec{0}\}$ in the range $\mathbb{R}^n$, what subspace would $\text{PreIm}(V)$ be? No explanation necessary.

$\ker(T)$ ✓

2 (d) [2 pts] Circle the one correct statement out of the four choices below (hint: think about $T : \text{PreIm}(V) \to V$ rather than $T : \mathbb{R}^m \to \mathbb{R}^n$).

PreIm(V) is all vector $\vec{x}$ in $\mathbb{R}^m$ s.t. $T(\vec{x})$ is in V in $\mathbb{R}^n$

- $\dim(\text{PreIm}(V)) = \dim(V)$ regardless of V and/or T.
- $\dim(\text{PreIm}(V)) \geq \dim(V)$ regardless of V and/or T.
- $\dim(\text{PreIm}(V)) \leq \dim(V)$ regardless of V and/or T. ✓
- In some cases $\dim(\text{PreIm}(V)) > \dim(V)$. In other cases $\dim(\text{PreIm}(V)) < \dim(V)$. Finally, there are also cases where $\dim(\text{PreIm}(V)) = \dim(V)$.

3. Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ be a basis for $\mathbb{R}^2$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ be two linear transformations that treat this basis in the following manner:

$$T(\vec{v}_1) = \vec{v}_2, \quad T(\vec{v}_2) = \vec{v}_1, \quad S(\vec{v}_1) = \vec{v}_1 - \vec{v}_2, \quad S(\vec{v}_2) = \vec{v}_2$$

4

(a) [4 pts] Write down the matrix B for the composition $T \circ S$ in $\mathcal{B}$-coordinates.

$[\vec{v}_1]_{\mathcal{B}} = \begin{bmatrix}1\\0\end{bmatrix}$

$[\vec{v}_2]_{\mathcal{B}} = \begin{bmatrix}0\\1\end{bmatrix}$

$T\circ S(\vec{v}_1) = T(S(\vec{v}_1)) = T(\vec{v}_1 - \vec{v}_2) = T(\vec{v}_1) - T(\vec{v}_2) = \vec{v}_2 - \vec{v}_1$

$[T\circ S(\vec{v}_1)]_{\mathcal{B}} = \begin{bmatrix}0\\1\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix}$

$T(S(\vec{v}_1)) = \vec{v}_2 - \vec{v}_1 = \begin{bmatrix}0\\1\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix}$

$T\circ S(\vec{v}_2) = T(S(\vec{v}_2)) = T(\vec{v}_2) = \vec{v}_1$

$[T\circ S(\vec{v}_2)]_{\mathcal{B}} = \begin{bmatrix}1\\0\end{bmatrix}$

$T(S(\vec{v}_2)) = \begin{bmatrix}1\\0\end{bmatrix}$

$$B = \begin{bmatrix} [T\circ S(\vec{v}_1)]_{\mathcal{B}} & [T\circ S(\vec{v}_2)]_{\mathcal{B}} \end{bmatrix} = \boxed{\begin{bmatrix}-1 & 1\\ 1 & 0\end{bmatrix}}$$

6

(b) [6 pts] Prove that, if we assume that S was an orthogonal transformation, then the basis vectors $\vec{v}_1$ and $\vec{v}_2$ *cannot* be orthogonal to each other.

if S is orthogonal transf, $S(\vec{v}_1) \cdot S(\vec{v}_2) = \vec{v}_1 \cdot \vec{v}_2$

if $\vec{v}_1$ and $\vec{v}_2$ are orthogonal to each other,

$\vec{v}_1 \cdot \vec{v}_2 = 0$.

$S(\vec{v}_1) = \vec{v}_1 - \vec{v}_2$ $\quad (\vec{v}_1 - \vec{v}_2) \cdot \vec{v}_2 = \vec{v}_1 \cdot \vec{v}_2 - \vec{v}_2 \cdot \vec{v}_2$

$S(\vec{v}_2) = \vec{v}_2$ $\quad = 0 - \|\vec{v}_2\|^2$

if $\vec{v}_1 \cdot \vec{v}_2 = S(\vec{v}_1) \cdot S(\vec{v}_2)$, $0 = 0 - \|\vec{v}_2\|^2$ and

$\vec{v}_2 = \vec{0}$, but $\vec{v}_1$ and $\vec{v}_2$ make up a basis

so $\vec{v}_2 \neq \vec{0}$, so $\vec{v}_1$ and $\vec{v}_2$ cannot be orthogonal to each other if S is orthogonal.

4. Multiple choice and/or true and false (circle one answer only; no justification needed). In all of the problems below, A is an nxm matrix.

(a) [2 pts] What can we say about $\dim(\operatorname{im}(A)) + \dim(\ker(A))$?

Always $= n$ Always $= m$ Neither of these

(b) [2 pts] What can we say about $\dim(\ker(A))$?

Always $< n$ Always $> n$ Neither of these

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

nullity(A) + rank(A) = m

rank(A) ≤ m
≤ n

nullity(A) = m − rank(A)
≥ m−n

− rank(A) ≥ −n.

if m ≤ n, nullity(A) = 0
m > n, nullity(A) ≥ m−n

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$x_1 \; x_2 \; x_3 \; x_4$

$x_2 = t$
$x_3 = s$
$x_4 = r$

$x_1 = -t - s - r$

(c) [2 pts] If B is a pxn matrix, then we must have $\operatorname{rank}(BA) \leq \operatorname{rank}(A)$.

TRUE FALSE

BA
pxn nxm
pxm

rank ≤ m

B A

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$\mathbb{R}^m \to \mathbb{R}^n \to \mathbb{R}^p$