MATH 33A | Midterm 2 | Feb. 25, 2019

STUDENT NAME: [redacted]

STUDENT ID NUMBER: [redacted]

DISCUSSION SECTION NUMBER: 3F

Directions

Answer each question in the space provided. Please write clearly and legibly. Show all of your work—your work must both justify and clearly identify your final answer. No books, notes or calculators are allowed. You must simplify results of function evaluations when it is possible to do so.

For instructor use only

Page	Points	Score
2	8	8
3	8	8
4	4	4
5	8	8
6	6	4
7	10	8
8	6	6
Total:	50	46

1. Consider the matrix

$$A = \begin{pmatrix} 1 & 5 & -3 \\ 2 & 0 & 4 \\ 2 & 2 & 2 \end{pmatrix}.$$

4 (a) [4 pts] Find a basis for ker(A), which is one-dimensional.

Solve $A\vec{x}=0$:

$$\left(\begin{array}{ccc|c} 1 & 5 & -3 & 0 \\ 2 & 0 & 4 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right) \begin{array}{l} \\ -2(\mathrm{I}) \\ -2(\mathrm{I}) \end{array} \longrightarrow \left(\begin{array}{ccc|c} 1 & 5 & -3 & 0 \\ 0 & -10 & 10 & 0 \\ 0 & -8 & 8 & 0 \end{array}\right) \begin{array}{l} \\ /-10 \\ \\ \end{array} \longrightarrow \left(\begin{array}{ccc|c} 1 & 5 & -3 & 0 \\ 0 & +1 & -1 & 0 \\ 0 & -8 & 8 & 0 \end{array}\right) \begin{array}{l} -5(\mathrm{II}) \\ \\ +8(\mathrm{II}) \end{array} \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $r=x_3$. Then: $x_1+2r=0 \iff x_1=-2r$.

$x_2-r=0 \iff x_2=r$.

$$\vec{x}=\begin{pmatrix} -2r \\ r \\ r \end{pmatrix} = r\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \iff \text{basis of ker}(A) \text{ is } \boxed{\left\{\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}\right\}}$$

2 (b) [2 pts] Using your kernel computation, or otherwise, write down a linear relation between the columns of A that shows that the third column can be viewed as redundant. No further explanation necessary.

$$\begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} - 1\begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}. \iff \vec{v}_3 = 2\vec{v}_2 - \vec{v}_1$$

if $A=\begin{pmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{pmatrix}$.

2 (c) [2 pts] Write down a basis for im(A). You may use any of the previous parts of the problem with no further explanation, even if you couldn't solve them. Or you may solve this problem from scratch using row reduction.

From part (b), $\begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$ is redundant, so:

basis of im(A) is $\boxed{\left\{\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}\right\}}$

(d) [8 pts] Convert your answer from the previous part of the problem into an orthonormal basis for im(A). (Keep your work well-organized!)

$$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{\sqrt{1^2+2^2+2^2}}\begin{pmatrix}1\\2\\2\end{pmatrix} = \begin{pmatrix}1/3\\2/3\\2/3\end{pmatrix}.$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{u}_1\cdot\vec{v}_2)\vec{u}_1 = \begin{pmatrix}5\\0\\2\end{pmatrix} - \left(5\left(\frac{1}{3}\right)+0\left(\frac{2}{3}\right)+2\left(\frac{2}{3}\right)\right)\begin{pmatrix}1/3\\2/3\\2/3\end{pmatrix}$$

$$= \begin{pmatrix}5\\0\\2\end{pmatrix} - 3\begin{pmatrix}1/3\\2/3\\2/3\end{pmatrix} = \begin{pmatrix}5-1\\0-2\\2-2\end{pmatrix} = \begin{pmatrix}4\\-2\\0\end{pmatrix}.$$

$$\vec{u}_2 = \frac{1}{\|\vec{v}_2^{\perp}\|}\vec{v}_2^{\perp} = \frac{1}{\sqrt{4^2+(-2)^2+0^2}}\begin{pmatrix}4\\-2\\0\end{pmatrix} = \frac{1}{\sqrt{20}}\begin{pmatrix}4\\-2\\0\end{pmatrix} = \frac{1}{2\sqrt{5}}\begin{pmatrix}4\\-2\\0\end{pmatrix} = \begin{pmatrix}2/\sqrt{5}\\-1/\sqrt{5}\\0\end{pmatrix}.$$

orthonormal basis of im(A) is $\boxed{\left\{\begin{pmatrix}1/3\\2/3\\2/3\end{pmatrix}, \begin{pmatrix}2/\sqrt{5}\\-1/\sqrt{5}\\0\end{pmatrix}\right\}}$

(e) [4 pts] Use your work from the previous part of the problem to write a $M = QR$ factorization for a relevant matrix M. (Part of the problem is deciding what M should be!)

$$M = \begin{pmatrix} 1 & 5 \\ 2 & 0 \\ 2 & 2 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 1/3 & 2/\sqrt{5} \\ 2/3 & -1/\sqrt{5} \\ 2/3 & 0 \end{pmatrix} \begin{pmatrix} 3 & 3 \\ 0 & 2\sqrt{5} \end{pmatrix}} \checkmark$$

Note that

$\vec{u}_1 = \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$, $\vec{u}_2 = \begin{pmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \\ 0 \end{pmatrix}$

from (d).

$$\|\vec{v}_1\| = \sqrt{1^2 + 2^2 + 2^2} = 3$$

$$\|\vec{v}_2^{\perp}\| = \sqrt{4^2 + 0^2 + (-2)^2} = \sqrt{20} = 2\sqrt{5}$$

$$\vec{u}_1 \cdot \vec{v}_2 = \frac{1}{3}(5) + \frac{2}{3}(0) + \frac{2}{3}(2) = 3.$$

2. Suppose we have a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$, and suppose further that we have a subspace V in the range $\mathbb{R}^n$. We will define a new subset of the *domain* $\mathbb{R}^m$ called the *pre-image* of V as follows:

$$\mathrm{PreIm}(V) := \{\vec{x} \text{ in } \mathbb{R}^m \mid T(\vec{x}) \text{ is in } V\}.$$

That is, $\mathrm{PreIm}(V)$ consists of all vectors in $\mathbb{R}^m$ that get mapped by T to a vector that is in V.

(a) [8 pts] Prove that $\mathrm{PreIm}(V)$ is a subspace of the domain $\mathbb{R}^m$ by checking all three necessary conditions.

Condition 1: $\vec{0} \in \mathrm{PreIm}(V)$

Since T is a linear transformation, then $T(\vec{0}) = \vec{0}$, and $\vec{0} \in V$ since V is a subspace. $\Leftrightarrow \vec{0} \in \mathrm{PreIm}(V)$ ✓.

Condition 2: $\vec{v} + \vec{w} \in \mathrm{PreIm}(V)$ if $\vec{v} \in \mathrm{PreIm}(V)$ and $\vec{w} \in \mathrm{PreIm}(V)$, (for any vectors $\vec{v}, \vec{w}$).

Since T is a linear transformation, $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$.

$\vec{v} \in \mathrm{PreIm}(V)$ implies $T(\vec{v}) \in V$ and $\vec{w} \in \mathrm{PreIm}(V)$ implies $T(\vec{w}) \in V$.

Thus, $T(\vec{v}) + T(\vec{w}) \in V$ by definition of the subspace and $T(\vec{v}+\vec{w}) \in V \Leftrightarrow \vec{v}+\vec{w} \in \mathrm{PreIm}(V)$ ✓

Condition 3: If $\vec{v} \in \mathrm{PreIm}(V)$, $k\vec{v} \in \mathrm{PreIm}(V)$ for any scalar k, and vector $\vec{v}$.

Since T is a linear transformation, $T(k\vec{v}) = kT(\vec{v})$.

$\vec{v} \in \mathrm{PreIm}(V)$ implies $T(\vec{v}) \in V$,

and by definition of the subspace, $k \cdot T(\vec{v}) \in V$.

$\Leftrightarrow T(k\vec{v}) \in V \Leftrightarrow k\vec{v} \in \mathrm{PreIm}(V)$. ✓

2 (b) [2 pts] If we had chosen the subspace V to be all of the range $\mathbb{R}^n$, what subspace would $\text{PreIm}(V)$ be? No explanation necessary.

$\mathbb{R}^m$

0 (c) [2 pts] If we had chosen the subspace V to be the zero subspace $\{\vec{0}\}$ in the range $\mathbb{R}^n$, what subspace would $\text{PreIm}(V)$ be? No explanation necessary.

$\vec{0}$

2 (d) [2 pts] Circle the one correct statement out of the four choices below (hint: think about $T : \text{PreIm}(V) \to V$ rather than $T : \mathbb{R}^m \to \mathbb{R}^n$).

- $\dim(\text{PreIm}(V)) = \dim(V)$ regardless of V and/or T.
- $\dim(\text{PreIm}(V)) \geq \dim(V)$ regardless of V and/or T.
- $\dim(\text{PreIm}(V)) \leq \dim(V)$ regardless of V and/or T.
- In some cases $\dim(\text{PreIm}(V)) > \dim(V)$. In other cases $\dim(\text{PreIm}(V)) < \dim(V)$. Finally, there are also cases where $\dim(\text{PreIm}(V)) = \dim(V)$.

3. Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ be a basis for $\mathbb{R}^2$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ be two linear transformations that treat this basis in the following manner:

$$T(\vec{v}_1) = \vec{v}_2, \quad T(\vec{v}_2) = \vec{v}_1, \qquad S(\vec{v}_1) = \vec{v}_1 - \vec{v}_2, \quad S(\vec{v}_2) = \vec{v}_2$$

(a) [4 pts] Write down the matrix B for the composition $T \circ S$ in $\mathcal{B}$-coordinates.

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}}$$

(b) [6 pts] Prove that, if we assume that S was an orthogonal transformation, then the basis vectors $\vec{v}_1$ and $\vec{v}_2$ *cannot* be orthogonal to each other.

Assume by contradiction that $\vec{v}_1 \perp \vec{v}_2$. Then:

S was an orthogonal transformation $\Leftrightarrow$

$$(\vec{v}_1 - \vec{v}_2) \cdot \vec{v}_2 = 0$$

(why assume $\|v_2\| = 1$ -2

$$\Leftrightarrow \quad \vec{v}_1 \cdot \vec{v}_2 - \vec{v}_2 \cdot \vec{v}_2 = 0$$

$$\Leftrightarrow \quad 0 - 1 = 0 \qquad (\vec{v}_1 \cdot \vec{v}_2 = 0, \ \vec{v}_2 \cdot \vec{v}_2 = 1)$$

$$\Leftrightarrow \quad -1 = 0 \quad \text{※}.$$

Thus $\vec{v}_1$ isn't orthogonal to $\vec{v}_2$ if S is orthogonal.

4. Multiple choice and/or true and false (circle one answer only; no justification needed). In all of the problems below, A is an nxm matrix.

   (a) [2 pts] What can we say about $\dim(\text{im}(A)) + \dim(\ker(A))$?

   Always $= n$ Always $= m$ Neither of these

   (b) [2 pts] What can we say about $\dim(\ker(A))$?

   Always $< n$ Always $> n$ Neither of these

   (c) [2 pts] If B is a pxn matrix, then we must have $\text{rank}(BA) \leq \text{rank}(A)$.

   TRUE FALSE