1. [6 pts] Is the vector $\vec{b} = \begin{pmatrix} -4 \\ -3 \\ 2 \end{pmatrix}$ a linear combination of the vectors $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$? If so, write down the linear combination in the format $\vec{b} = c_1\vec{v} + c_2\vec{w}$. If not, explain why not.

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 3 & -4 \\ 0 & 1 & -3 \\ 1 & 1 & 2 \end{array}\right] \begin{matrix} \\ \\ -\mathrm{I} \end{matrix} \longrightarrow \left[\begin{array}{cc|c} 1 & 3 & -4 \\ 0 & 1 & -3 \\ 0 & -2 & 6 \end{array}\right] \begin{matrix} -3\mathrm{II} \\ \\ +2\mathrm{II} \end{matrix} \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

yes.

$$\begin{bmatrix} -4 \\ -3 \\ 2 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$5 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix} - \begin{bmatrix} 9 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \\ 2 \end{bmatrix}$$

2. [5 pts] Consider the following matrix:

$$A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 3 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}$$

$A\vec{x} = 0$

Circle all of the following vectors which are members of ker(A).

$$\underbrace{\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{w} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \quad \vec{z} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}}_{\text{general } \vec{u}}$$

$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 3 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix} \vec{u} = \vec{0}$$

$$u_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \\ 1 \end{bmatrix} + u_2 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 4 \\ 1 \end{bmatrix} + u_3 \begin{bmatrix} -1 \\ 1 \\ 0 \\ -3 \\ -1 \end{bmatrix} + u_4 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \vec{0}$$

$\vec{u} = \vec{v}$ $= \begin{bmatrix} 3 \\ \vdots \end{bmatrix} \neq \vec{0}$

$\vec{u} = \vec{w} = \vec{0}$ is always in ker(A)

$$\vec{u} = \vec{x} = \begin{bmatrix} 2 \\ -2 \\ 0 \\ 6 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 0 \\ -6 \\ -2 \end{bmatrix} = \vec{0}$$

$$\vec{u} = \vec{y} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 0 \\ -6 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ \vdots \end{bmatrix} \neq \vec{0}$$

$$\vec{u} = \vec{z} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 0 \\ -4 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$$

3. Suppose you know that $A^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $B^{-1} = \begin{pmatrix} -1 & 1 \\ 3 & 2 \end{pmatrix}$.

6 (a) [6 pts] Find $(AB)^{-1}$.

$(AB)^{-1} = B^{-1}A^{-1}$

$$\begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -2+1 & -1+1 \\ 6+2 & 3+2 \end{bmatrix} = \boxed{\begin{bmatrix} -1 & 0 \\ 8 & 5 \end{bmatrix}}$$ ✓

$AB\vec{x} = \vec{b}$
$B\vec{x} = A^{-1}\vec{b}$
$\vec{x} = B^{-1}A^{-1}\vec{b}$

$$\begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} -2 & -1 \\ 6 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 0 \\ 8 & 5 \end{bmatrix}$$

5 (b) [5 pts] Find B.

$BB^{-1} = I_2$ $\qquad (B^{-1} \,\vdots\, I_2) \Rightarrow (I_2 \,\vdots\, B)$

$$\left[\begin{array}{cc|cc} -1 & 1 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right] \xrightarrow{\times -1} \left[\begin{array}{cc|cc} 1 & -1 & -1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right] \xrightarrow{-3\mathrm{I}} \left[\begin{array}{cc|cc} 1 & -1 & -1 & 0 \\ 0 & 5 & 3 & 1 \end{array}\right] \xrightarrow{\div 5}$$

$$\left[\begin{array}{cc|cc} 1 & -1 & -1 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{1}{5} \end{array}\right] \xrightarrow{+\mathrm{II}} \left[\begin{array}{cc|cc} 1 & 0 & -\frac{2}{5} & \frac{1}{5} \\ 0 & 1 & \frac{3}{5} & \frac{1}{5} \end{array}\right]$$

$$\boxed{B = \begin{bmatrix} -\frac{2}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{1}{5} \end{bmatrix}}$$ ✓

2 (c) [2 pts] What was the rank of A? (this should require no computations) ✓

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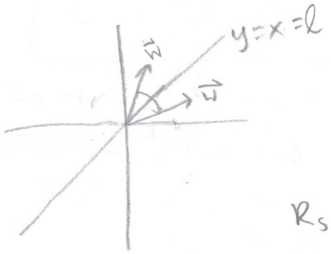
4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that rotates all vectors about the origin *counter-clockwise* by $\pi/2$. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that reflects all vectors about the line $y = x$.

6

(a) [6 pts] Find the single matrix representing the composite function $T \circ \overset{S}{R}$. *Hint: A geometric approach may be easier than a computation.*



$\text{Proj}_\ell(\vec{x}) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ $\quad \vec{u} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

$A_T = \begin{bmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$R_S = 2P - I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

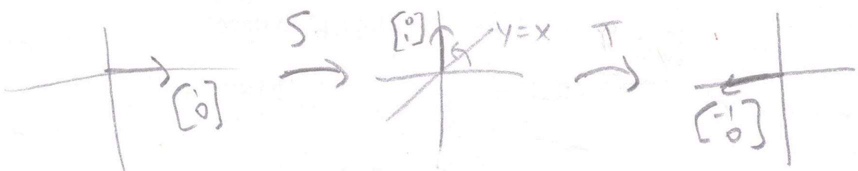
R_S is reflection matrix for S

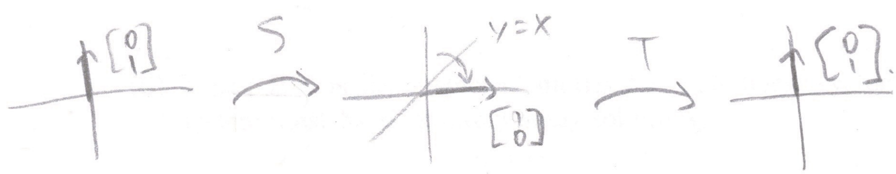
A_T is matrix for T

$T \circ S = A_T R_S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\boxed{M_{T\circ S} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$M_{T\circ S}$ is matrix of $T\circ S$





2

(b) [2 pts] Find $T(\overset{S}{R}(\vec{v}))$ where $\vec{v} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

$T(S(\vec{v})) = M_{T\circ S}\begin{pmatrix} -2 \\ 3 \end{pmatrix} \quad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$$\boxed{T(S(\vec{v})) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}}$$

(c) [6 pts] Find the single matrix representing the composite function $R \circ T$. *Hint: A geometric approach may be easier than a computation.*

1 → T rotates CCW by $\frac{\pi}{2}$

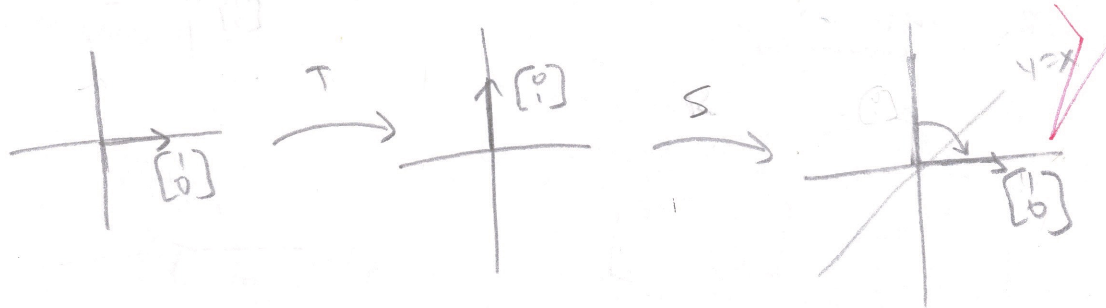
2 → S reflects over $y=x$

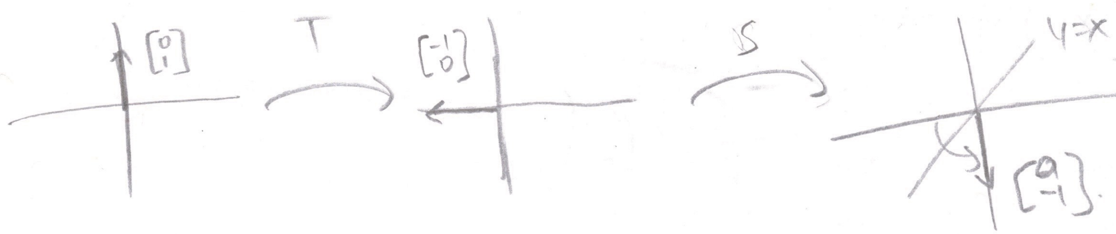
$S \circ T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix}$

$S \circ T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\-1\end{bmatrix}$

$$\boxed{M_{S\circ T} = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}}$$

$M_{S\circ T}$ is matrix of composite function $R \circ T$





(d) [2 pts] Find $R\left(T\left(\vec{v}\right)\right)$ where $\vec{v} = \begin{pmatrix}-2\\3\end{pmatrix}$.

$S(T(\vec{v})) = M_{S\circ T}\left(\begin{bmatrix}-2\\3\end{bmatrix}\right) = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}-2\\3\end{bmatrix}$

$= \begin{bmatrix}-2\\-3\end{bmatrix}$

$$\boxed{S(T(\vec{v})) = \begin{pmatrix}-2\\-3\end{pmatrix}}$$

5. [4 pts] Suppose you know that $\vec{w}$ is in ker(B), and you also know that $B\vec{v} = \begin{pmatrix}1\\2\\3\end{pmatrix}$. Use this information to find $B(2\vec{w} - 3\vec{v})$.

$B\vec{w} = \vec{0}$

$$B(2\vec{w} - 3\vec{v}) = 2B(\vec{w}) - 3B(\vec{v})$$
$$= 2(\vec{0}) - 3\begin{bmatrix}1\\2\\3\end{bmatrix}$$

$$\boxed{B(2\vec{w} - 3\vec{v}) = \begin{bmatrix}-3\\-6\\-9\end{bmatrix}}$$

6. True or false (circle your answer; no justification needed). In all of the problems below, A is an nxn *square* matrix.

(a) [2 pts] If A is the coefficient matrix for some linear system, and rank$(A) = n$, then the system has a unique solution.

TRUE FALSE

(b) [2 pts] If A is the coefficient matrix for some linear system, and rank$(A) < n$, then the system must have infinitely many solutions.

TRUE FALSE

(c) [2 pts] If A is the coefficient matrix for some linear system, and the system had a unique solution, then the RREF of A must be precisely the identity matrix $I_n = \begin{pmatrix}1 & 0 & 0 & \cdots & 0\\0 & 1 & 0 & & 0\\0 & 0 & 1 & & 0\\\vdots & & & \ddots & \\0 & 0 & 0 & \cdots & 1\end{pmatrix}$.

TRUE FALSE