Math 33A/1
Spring 2016
04/22/16
Time Limit: 50 Minutes

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SID Number: 704 -

Day \ T.A.	David	Casey	Adam
Tuesday	1A	1C	1E
Thursday	1B	1D	1F

This exam contains 5 pages (including this cover page) and 4 problems. Check t are missing. Enter your name and SID number on the top of this page, cross the b to your discussion section, and put your initials on the top of every page, in case the pages become separated. Also, have your photo ID on the desk in front of you during the exam.

Use a pen to record your answers. Calculators or computers of any kind are not allowed. You are not allowed to consult any other materials of any kind, including books, notes and your neighbors. You may use the back of this sheet for your notes ("scratch paper"). If you need additional paper, let the proctors know.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	10	6
2	10	10
3	10	10
4	10	10
Total:	40	36

Of course, if you have a question about a particular problem, please raise your hand and one of the proctors will come and talk to you.

☆ answer on back ☆

S.G.

1. (10 points) Consider the following system of linear equations:

$$\left|\begin{aligned} 2x - 4y + z &= 0 \\ x + ky &= 0 \\ 2y + kz &= 1 \end{aligned}\right|,$$

$\frac{1}{2} + (k+2)\left(\frac{k}{2}\right) = \frac{1}{2} + \frac{k^2}{2} + \frac{2k}{2}$

where k is a real constant.

(a) For which values of k does the system have a unique solution? No solutions? Infinitely many?

(b) Solve the system when $k = 0$.

a) $\left[\begin{array}{ccc|c} 2 & -4 & 1 & 0 \\ 1 & k & 0 & 0 \\ 0 & 2 & k & 1 \end{array}\right] :2 \Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & \frac{1}{2} & 0 \\ 1 & k & 0 & 0 \\ 0 & 2 & k & 1 \end{array}\right] -(\mathrm{I}) = \left[\begin{array}{ccc|c} 1 & -2 & \frac{1}{2} & 0 \\ 0 & k+2 & -\frac{1}{2} & 0 \\ 0 & 2 & k & 1 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & -2 & \frac{1}{2} & 0 \\ 0 & 2 & k & 1 \\ 0 & k+2 & -\frac{1}{2} & 0 \end{array}\right] :2 \Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{k}{2} & \frac{1}{2} \\ 0 & k+2 & +\frac{1}{2} & 0 \end{array}\right] \begin{array}{l} +2(\mathrm{II}) \\ \\ -(k+2)(\mathrm{II}) \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2}+k & 1 \\ 0 & 1 & \frac{k}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2}+\frac{k^2}{2}+\frac{2k}{2} & \frac{k}{2}+1 \end{array}\right]$

+3

$\frac{k^2}{2} + \frac{1}{2} + \frac{2k}{2} \to k^2 + 1 + 2k \Rightarrow (k+1)^2 \quad k = -1, \quad 0 = \frac{-1}{2} + 1 \quad 0 = -\frac{1}{2} \Rightarrow$ no sol.

$0 = 0$, so $\quad k^2 + 1 + 2k = k + 2 \quad (k^2 + k - 1) = 0 \rightarrow \frac{-1 \pm \sqrt{1 - 4(-1)}}{2} \rightarrow \frac{-1 \pm \sqrt{5}}{2}$

∞ solutions when $\boxed{k = \frac{-1 \pm \sqrt{5}}{2}}$

$\frac{1}{2} + 2 + \frac{-4}{2} = -1$

$\frac{1}{2} = -\frac{1}{2}$

exactly one solution for anything else

b) $\left[\begin{array}{ccc|c} 2 & -4 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right] :2 \quad \left[\begin{array}{ccc|c} 1 & -2 & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right] -(\mathrm{I}) = \left[\begin{array}{ccc|c} 1 & -2 & \frac{1}{2} & 0 \\ 0 & 2 & -\frac{1}{2} & 0 \\ 0 & 2 & 0 & 1 \end{array}\right] \begin{array}{l} +\mathrm{II} \\ \\ -\mathrm{II} \end{array}$

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 2 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 1 \end{array}\right] + \mathrm{III} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & 1 \end{array}\right] \begin{array}{l} \\ :2 \\ :\frac{1}{2} \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 2 \end{array}\right] \rightarrow \boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 2 \end{bmatrix}}$

+2

a) No solution: $k = -1$

Infinite solutions: $k = \frac{-1+\sqrt{5}}{2}$, $k = \frac{-1-\sqrt{5}}{2}$

1 solution: $k \neq -1, \frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}$

2. (10 points) (a) Find the inverse of the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 3 & 0 & 0 \end{bmatrix}$.

$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 3 & 0 & 0 \end{pmatrix} \cdot \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ -2 & 1 & 0 \\ 1 & 0 & -\frac{1}{3} \end{bmatrix} = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$ ✓

(b) Find a 4×3 matrix A satisfying

$$A\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\2\\0\\4\end{bmatrix}, \qquad A\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\1\\-1\\-2\end{bmatrix}, \qquad A\begin{bmatrix}1\\0\\2\end{bmatrix} = \begin{bmatrix}-3\\2\\-1\\1\end{bmatrix}.$$

(Hint: Notice that e.g. $\begin{bmatrix}1\\0\\0\end{bmatrix} = 2\begin{bmatrix}1\\1\\1\end{bmatrix} - 2\begin{bmatrix}0\\1\\0\end{bmatrix} - \begin{bmatrix}1\\0\\2\end{bmatrix}$*.)*

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a) $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 3 & 0 & 0 \end{bmatrix} \begin{matrix} \\ -2(I) \\ -3(I) \end{matrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{matrix} \\ \\ :-3 \end{matrix} \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - III = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$ ✓ invertible

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \\ -2(I) \\ -3(I) \end{matrix} = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array}\right] \begin{matrix} \\ \\ \div -3 \end{matrix} = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right] - III$

$\left.\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array}\right. \to \boxed{\begin{bmatrix} 0 & 0 & \frac{1}{3} \\ -2 & 1 & 0 \\ 1 & 0 & -\frac{1}{3} \end{bmatrix}}$

6

b) 4x3 satisfying that $\to \overset{3}{\begin{bmatrix} \quad \end{bmatrix}}$

$A\begin{bmatrix}0\\1\\0\end{bmatrix} \Rightarrow \begin{bmatrix} - & 0 & - \\ - & 1 & - \\ - & -1 & - \\ - & -2 & - \end{bmatrix} \to 1\begin{bmatrix}a\\b\\c\\d\end{bmatrix} + 2\begin{bmatrix}e\\f\\g\\h\end{bmatrix} = \begin{bmatrix}-3\\2\\-1\\1\end{bmatrix} \to 1\begin{bmatrix}a\\b\\c\\d\end{bmatrix} + 1\begin{bmatrix}e\\f\\g\\h\end{bmatrix} = \begin{bmatrix}1\\1\\1\\6\end{bmatrix}$

$\to$ $a+2e=-3$ $\quad a+e=1$ $\quad a=1-e$ $\to$ $1-e+2e=-3$ $\quad e=-4$ $\quad a=5$

$b+2f=2, \quad b+f=1 \quad b=1-f \to 1-f+2f=2 \quad f=1, \quad b=1-1=0$

$c+2g=-1 \quad c+g=1 \quad c=1-g \quad 1-g+2g=-1 \quad 1+g=-1 \quad g=-2, \quad c=1-(-2)=3$

$d+2h=1 \quad d+h=6 \quad d=6-h \quad 6-h+2h=1 \quad 6+h=1 \quad h=-5 \quad d=6-(-5)=11$

$\to \boxed{A = \begin{bmatrix} 5 & 0 & -4 \\ 0 & 1 & 1 \\ 3 & -1 & -2 \\ 11 & -2 & -5 \end{bmatrix}}$

S.G.

3. (10 points) (a) Find the matrix of reflection about the line $4y = 3x$ in $\mathbb{R}^2$.

(b) Describe the kernel of this matrix geometrically.

(c) Is the set of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ satisfying $4y = 3x^2$ a subspace of $\mathbb{R}^2$? Justify your answer.

a) Reflection about line $y = \frac{3}{4}x \rightarrow$ [redacted] $\begin{bmatrix} 4 \\ 3 \end{bmatrix} \rightarrow$ 3, 4 $\rightarrow y = \frac{3}{4}x$ ✓

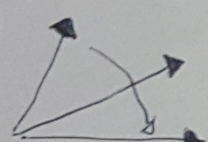
~~Orthogonal projection~~ $\vec{u} = \frac{1}{5} \cdot \frac{4}{3} = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix} \rightarrow proj_L(\vec{x}) = \begin{bmatrix} \frac{16}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{9}{25} \end{bmatrix}$

reflection $= 2proj_L(\vec{x}) - I = \begin{bmatrix} \frac{32}{25} - 1 & \frac{24}{25} \\ \frac{24}{25} & \frac{18}{25} - 1 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{7}{25} & \frac{24}{25} \\ \frac{24}{25} & \frac{-7}{25} \end{bmatrix}}$ ✓

Check: $\left(\frac{7}{25}\right)^2 + \left(\frac{24}{25}\right)^2 = \left(\frac{25}{25}\right)^2$ ✓

$= 7, 24, 25$ triangle !!

b)



Kernel is $A \cdot \vec{x} = 0 \rightarrow \frac{1}{25}\begin{bmatrix} 7 & 24 \\ 24 & -7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow$ just $\vec{0}$

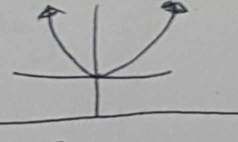
The kernel of this matrix is just the 0 vector, $\vec{0}$, which means geometrically that ~~there~~ is no relation among the columns of the matrix and only the 0 vector will result in $A\vec{x} = \vec{0}$ to be true.

invertible

★★ No combination of the columns in this vector will result in $\vec{0}$. ★★

6

c) $\begin{bmatrix} x \\ y \end{bmatrix}$ $4y = 3x^2 \rightarrow y = \frac{3}{4}x^2$



No, this is not a subspace of $\mathbb{R}^2$ as multiplying by a negative scalar such as $k = -1$ does not hold true with this function $(y = \frac{3}{4}x^2)$. A subspace must be closed under scalar multiplication!

S.G.

4. (10 points) (a) Identify the redundant vectors in the following list:

$$\begin{bmatrix}-1\\1\\3\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\-2\\-3\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\2\\4\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\times \quad \times \quad \times$$

(b) Consider the matrix $A = \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}$. Compute A^{2016}.

a) 6

$$\begin{bmatrix}-1 & 0 & -1 & 1 & 0 & 0\\ 1 & 0 & -2 & 1 & 2 & 0\\ 3 & 0 & -3 & 1 & 4 & 1\end{bmatrix} \begin{matrix} \\ +(I) \\ +3(I)\end{matrix} \Rightarrow \begin{bmatrix}-1 & 0 & -1 & 1 & 0 & 0\\ 0 & 0 & -3 & 2 & 2 & 0\\ 0 & 0 & -6 & 4 & 4 & 1\end{bmatrix} : -3$$

$$\begin{bmatrix}-1 & 0 & -1 & 1 & 0 & 0\\ 0 & 0 & 1 & -\frac{2}{3} & -\frac{2}{3} & 0\\ 0 & 0 & -6 & 4 & 4 & 1\end{bmatrix} \begin{matrix} +II \\ \\ +6(II)\end{matrix} = \begin{bmatrix}-1 & 0 & 0 & \frac{1}{3} & -\frac{2}{3} & 0\\ 0 & 0 & 1 & -\frac{2}{3} & \frac{2}{3} & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}$$

$6 \cdot -\frac{2}{3} = -4$

The 3 redundant vectors are: $\boxed{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\2\\4\end{bmatrix}}$

b) $A = \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix} \rightarrow A^2 = \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix} \cdot \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix} = \begin{bmatrix}1 & 3\\0 & 4\end{bmatrix} \cdot \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix} = A^3 = \begin{bmatrix}1 & 7\\0 & 8\end{bmatrix}$

4 $A^4 = \begin{bmatrix}1 & 7\\0 & 8\end{bmatrix} \cdot \begin{bmatrix}1 & 1\\0 & 2\end{bmatrix} = \begin{bmatrix}1 & 15\\0 & 16\end{bmatrix}$

$\rightarrow$ general formula $\begin{bmatrix}1 & ?\\0 & ?\end{bmatrix} \rightarrow$ $2^4 = 2 \cdot 2 \cdot 2 \cdot 2$ ($4 \rightarrow 8 = 16$), $2^3 = 8$, $2^2 = 4 \rightarrow$ $2^n - 1$, 2^n

$$A^{2016} = \boxed{\begin{bmatrix}1 & 2^{2016} - 1\\0 & 2^{2016}\end{bmatrix}}$$