Math 33A, Sec 3
Linear Algebra and Applications
J. Madrid

Midterm 2. May 22-23, 2020

Instructions: You have 2 **hours** to complete this exam, in any subinterval **from Friday May 22 at 8:00 am to Saturday May 23 at 8:00 am, Pacific time**. There are five problems, worth a total of 25 points. This test is OPEN book and OPEN notes. Calculators are allowed.

For full credit show all of your work legibly and **justify all your answers!** (except in problem 1 (True or false question)).

Please write your solutions in white paper, take photos of your solution, put all of them in a single file and convert to pdf, then upload to CCLE before the two hours deadline. Then you have to submit your pdf file to gradescope from Saturday, May 23 at 8:00 am to Sunday May 24 at 8:00 am. Make sure that your pdf file CONTAIN your name and UID number. You don't need to attach your scratch work. Please **circle or box your final answers**.

Important information you should include in your pdf file:

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Question	Points	Score
1	3	
2	5	
3	6	
4	6	
5	5	
Total:	25	

Problem 2. *5pts.*

Consider the matrix $B = \begin{bmatrix} -5 & 12 \\ 12 & 5 \end{bmatrix}$ This is a reflection through a line L combined with a scaling by a factor $k \in \mathbb{R}$.

i.) (3 points) Find the equation of the line L.

ii.) (2 point) Find the matrix representing the orthogonal projection map onto L.

i.)

$$\frac{1}{m^2+1}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$$

scaling is determinant $= -25 - 144 = -169$

$$\begin{bmatrix} \frac{5}{169} & \frac{12}{-169} \\ \frac{12}{-169} & \frac{5}{-169} \end{bmatrix} \qquad \frac{12}{-169} = 2m$$

$$\boxed{m = \frac{6}{-169}}$$

$$\boxed{y = \frac{6}{-169}x}$$

ii.)

$$\text{proj}_{[L]}(\vec{v}) = \frac{\vec{v}\cdot\vec{L}}{\vec{L}\cdot\vec{L}}\vec{L}$$

we choose arbitrary vector for $\vec{L} = \begin{bmatrix} 1 \\ \frac{6}{-169} \end{bmatrix}$

$\rightarrow$

$$\frac{\vec{v}\cdot\vec{l}}{\vec{l}\cdot\vec{l}}\vec{l} \qquad \frac{\vec{v}\cdot\vec{l}}{1+\left(\frac{6}{-169}\right)^2}\begin{pmatrix}1\\ \frac{6}{-169}\end{pmatrix} = \begin{bmatrix}a_1\\ a_2\end{bmatrix}$$

$$\frac{v_1 + \frac{6}{-169}v_2}{1+\left(\frac{6}{-169}\right)^2}$$

$$\begin{bmatrix} \frac{1}{1+\left(\frac{6}{-169}\right)^2} & \frac{\frac{6}{-169}}{1+\left(\frac{6}{-169}\right)^2} \\ \frac{6}{-169}\left(\frac{1}{1+\left(\frac{6}{-169}\right)^2}\right) & \frac{6}{-169}\left(\frac{\frac{6}{-169}}{1+\left(\frac{6}{-169}\right)^2}\right) \end{bmatrix}$$

Problem 1. *9pts.*
Indicate which of the following are true or false; no justification is required: *[1pt Each question]*

(a) All orthogonal projections are orthogonal linear transformations. F

(b) The matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ is a rotation in $\mathbb{R}^3$. T

(c) For all $n \times n$ matrix A we have that $det(A) = det(rref(A))$. T

...onsider the matrix $A = \begin{bmatrix} 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

Find the eigenvalues of A and basis for each of the eigenspaces of A.
Is A diagonalizable?.

ii.) (3 points) Consider the matrix $M = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 2 \end{bmatrix}$. For which real values a, b, c is M diagonalizable?

i.)

$$A - \lambda \cdot I = \begin{bmatrix} 0-\lambda & 4 & 0 & 2 \\ 0 & 2-\lambda & 0 & 3 \\ 0 & 0 & 0-\lambda & 5 \\ 0 & 0 & 0 & 2-\lambda \end{bmatrix} = 0$$

← to find where equal to 0, we do

$$\det(A - \lambda \cdot I) = (-1)^0 \cdot -\lambda \cdot 2-\lambda \cdot -\lambda \cdot 2-\lambda$$

$$= \lambda^2 (2-\lambda)^2$$

eigenvalues $\boxed{\lambda = 2}$

$$E_2 = \begin{bmatrix} -2 & 4 & 0 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & -2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \ker \begin{bmatrix} -2 & 4 & 0 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & -2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \text{span} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

It is not diagonalizable b/c the geometric multiplicities don't add up to 4 and an eigenbasis cannot be formed ii.)

$$M - \lambda I = \begin{bmatrix} 1-\lambda & a & b \\ 0 & 1-\lambda & c \\ 0 & 0 & 2 \end{bmatrix}$$

$$\det(A - \lambda I) = (1-\lambda)(1-\lambda)(2)$$

$$= 2(1-\lambda)^2$$

$$\lambda = 1$$

There is only 1 eigenspace associated with this matrix and thus it doesn't matter what a, b, and c are because an eigenbasis is impossible to form. Thus there are no values of a, b, and c such that M is diagonalizable.

Problem 4. *6pts.*

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & -1 \\ 1 & 3 & 1 \\ 1 & 4 & -1 \end{bmatrix}.$$

i.) (4 points) Find an orthonormal basis for $Im(A)$.

ii.) (2 points) Find an orthonormal basis for $Im(A)^{\perp}$.

i.)

$$\vec{u}_1 = \frac{(1\ 1\ 1\ 1)}{\|(1\ 1\ 1\ 1)\|} = \boxed{\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right)}$$

$$\vec{u}_2 = \frac{(1\ 2\ 3\ 4) - \text{proj}_{(1111)}(1\ 2\ 3\ 4)}{\|\text{numerator}\|}$$

$$= \frac{(1\ 2\ 3\ 4) - \left(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\right)\cdot(1\ 2\ 3\ 4)\left(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\right)}{\|\text{numerator}\|}$$

$$= \frac{(-1.5, -0.5, .5, 1.5)}{\|\text{numerator}\|} = \boxed{\frac{(-1.5, -0.5, 0.5, 1.5)}{\sqrt{5}}}$$

$$\vec{u}_3 = (3\ -1\ 1\ -1) - \left(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\right)\cdot(3\ -1\ 1\ -1)\left(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\right) - \frac{(-1.5, -0.5, 0.5, 1.5)}{\sqrt{5}}(3\ -1\ 1\ -1)\frac{(-1.5, -0.5, 0.5, 1.5)}{\sqrt{5}}$$

$$= \boxed{\underbrace{(0.4082, -0.8165, 0.4082, 0)}_{\text{used calculator.}}}$$

ii.)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 3 & -1 & 1 & -1 \end{bmatrix}$$

the basis from i.) can be transposed to get

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{-1.5}{\sqrt{5}} & \frac{-0.5}{\sqrt{5}} & \frac{0.5}{\sqrt{5}} & \frac{1.5}{\sqrt{5}} \\ 0.4082 & -0.8165 & 0.4082 & 0 \end{bmatrix}$$

or

$$\left\{ \begin{bmatrix} \frac{1}{2} \\ \frac{-1.5}{\sqrt{5}} \\ 0.4082 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{-0.5}{\sqrt{5}} \\ -0.8165 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{0.5}{\sqrt{5}} \\ 0.4082 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1.5}{\sqrt{5}} \\ 0 \end{bmatrix} \right\}$$

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Problem 5. *5pts.*

[i] (2 point) Consider an inconsisten system of linear equations $Ax = b$ where A is a 2020×2019 matrix. Assume that x^* is the least-squares solution of this system. Let Q be an orthogonal 2020×2020 matrix. Find the least-squares solution(s) of the linear system $QAx = Qb$.

[ii] (3 points) Consider an orthonormal basis $u_1, u_2, \ldots, u_{2020}$ in $\mathbb{R}^{2020}$. Find the least-squares solution(s) of the system

$$Ax = u_{2020}$$

where $A = [u_1 \ u_2 \ u_3 \ \ldots \ u_{2019}]$.

Simplify your answers as much as possible.

i.) Since Q is orthogonal, and Ax is a vector, and Qb is a vetor, the solution is the solutions of

$$Q^T Q A x = Q^T Q b .$$

Since Q is orthogonal, $Q^T = Q^{-1}$ and thus $Ax = b$ and the least squares solution is still x^*.

ii.) Since $\vec{u}_{2020}$ is orthonormal to all the vectors in A, x^* will be 0 since the relationship between $\vec{u}_{2020}$ and all the other vectors are orthogonal.