Math 33A, Sec 3
Linear Algebra and Applications
J. Madrid

Midterm 1. April 24-25, 2020

Instructions: You have 2 **hours** to complete this exam, in any subinterval **from Friday April 24 at 8:00 am to Saturday April 25 at 8:00 am, Pacific time**. There are five problems, worth a total of 25 points. This test is OPEN book and OPEN notes. Calculators are allowed.

For full credit show all of your work legibly and **justify all your answers!** (except in problem 1 (True or false question)).

Please write your solutions in white paper, take photos of your solution, put all of them in a single file and convert to pdf, then upload to CCLE and gradescope by the deadline: Saturday, April 25 at 8:00 am. Make sure that your pdf file CONTAIN your name and UID number. You don't need to attach your scratch work. Please **circle or box your final answers.**

Important information you should include in your pdf file:

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Question	Points	Score
1	4	
2	5	
3	5	
4	6	
5	5	
Total:	25	

Problem 1. *4pts.*

Indicate which of the following are true or false; no justification is required: *[1pt Each question]*

(a) If a 3x3 matrix A is invertible, then $rref(A) = I_3$. $\boxed{T}$

(b) The equation $(A+B)^2 = A^2 + 2AB + B^2$ is true for all 2x2 matrices A and B. $\boxed{F}$

(c) The set of vectors $\left(\begin{bmatrix} 0 \\ 0 \\ 2020 \end{bmatrix}, \begin{bmatrix} 0 \\ 2020 \\ 2020 \end{bmatrix}, \begin{bmatrix} 2020 \\ 2020 \\ 2020 \end{bmatrix} \right)$ is a basis for $\mathbb{R}^3$. $\boxed{T}$

(d) The equation $(A)(A^{-1}) = (A^{-1})(A)$ is true for all invertible 10x10 matrices A. $\boxed{T}$

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$+\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$ $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ 2?

$\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix}$ $\begin{matrix} 5 & 6 \\ 7 & 8 \end{matrix}$

$\begin{matrix} 116 & 144 \\ 180 & 224 \end{matrix}$

Problem 2. *5pts.*

Consider the following system of linear equations

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 4 & 0 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \\ b \end{bmatrix}$$

i.) (2 points) For which values $a, b \in \mathbb{R}$ does the above system have a unique solution?

ii.) (2 points) For which values $a, b \in \mathbb{R}$ the above system has infinitely many solutions?

iii.) (1 point) For which values $a, b \in \mathbb{R}$ the above system has no solutions?

i.)

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 8 \\ 0 & 3 & 6 & 9 \\ 4 & 0 & a & b \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 8 \\ 0 & 3 & 6 & 9 \\ 0 & -8 & a-12 & b-32 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 8 \\ 0 & 1 & 2 & 3 \\ 0 & -8 & a-12 & b-32 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & a+4 & b+8 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & \\ 0 & 0 & \frac{a+4}{a+4} & \frac{b+8}{a+4} \end{array}\right]$$

There is a unique solution when $a \neq -4$

ii) There are infinitely many solutions when $a = -4$ & $b = -8$

iii.) There are no solutions when $a = -4$ & $b \neq -8$

Problem 3. *5pts.*

Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + 6y + 12z \\ 3y + 6z \\ 5x + 5z \end{bmatrix}$$

a.) (2 points) Find the matrix A such that $T(\vec{v}) = A\vec{v}$ for all $\vec{v} \in \mathbb{R}^3$.

b.) (3 points) Is T invertible? If yes, find the inverse of T.

a.)

$$\begin{bmatrix} 2 & 6 & 12 \\ 0 & 3 & 6 \\ 5 & 0 & 5 \end{bmatrix}$$

b.)

$$\left[\begin{array}{ccc|ccc} 2 & 6 & 12 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 & 0 \\ 5 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \quad \left[\begin{array}{ccc|ccc} 1 & 3 & 6 & \frac{1}{2} & 0 & 0 \\ 0 & 3 & 6 & 0 & 1 & 0 \\ 5 & 0 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 6 & \frac{1}{2} & 0 & 0 \\ 0 & 3 & 6 & 0 & 1 & 0 \\ 0 & -15 & -25 & -\frac{5}{2} & 0 & 1 \end{array}\right] \quad \left[\begin{array}{ccc|ccc} 1 & 3 & 6 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 2 & 0 & \frac{1}{3} & 0 \\ 0 & -15 & -25 & -\frac{5}{2} & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -1 & 0 \\ 0 & 1 & 2 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 5 & -\frac{5}{2} & 5 & 1 \end{array}\right] \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -1 & 0 \\ 0 & 1 & 2 & 0 & \frac{5}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 1 & \frac{1}{5} \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -1 & 0 \\ 0 & 1 & 0 & 1 & -\frac{5}{3} & -\frac{2}{5} \\ 0 & 0 & 1 & -\frac{1}{2} & 1 & \frac{1}{5} \end{array}\right]$$

Yes it is invertible b/c Rank $(T) = 3$

and the inverse is

$$\begin{bmatrix} \frac{1}{2} & -1 & 0 \\ 1 & -\frac{5}{3} & -\frac{2}{5} \\ -\frac{1}{2} & 1 & \frac{1}{5} \end{bmatrix}$$

Problem 4. *6pts.*

Consider the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 2 \\ 0 & 3 & 1 & 3 \end{bmatrix}$

i.) (2 points) Find $rank(A)$ and $nullity(A)$.

ii.) (2 points) Find a basis for $Im(A)$.

iii.) (2 points) Find a basis for $Ker(A)$.

i.) $\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 2 \\ 0 & 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{3} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{1}{3} & -1 \\ 0 & 1 & \frac{1}{3} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow$ rank (A) is 2

for an $m \times n$ matrix rank (A) + nullity(A) = n
and thus since rank(A) = 2 nullity(A) = 2

$\boxed{\text{Nullity}(A) \text{ is } 2}$

ii.) $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$ span the basis of Im (A)
(first 2 cols b/c of rref(A))

iii.) $1x_1 + \frac{1}{3}x_3 - 1x_4 = 0$
$1x_2 + \frac{1}{3}x_3 + 1x_4 = 0$ } from rref (A)

$x_1 = \frac{1}{3}x_3 - x_4$

$x_2 = \frac{1}{3}x_3 + x_4$

$\begin{bmatrix} \frac{1}{3}x_3 - x_4 \\ \frac{1}{3}x_3 + x_4 \\ x_3 + 0x_4 \\ 0x_3 + x_4 \end{bmatrix}$ ↪

$$\begin{bmatrix} \frac{1}{3}x_3 \\ \frac{1}{3}x_3 \\ x_3 \\ 0x_3 \end{bmatrix} + \begin{bmatrix} -x_4 \\ x_4 \\ 0x_4 \\ x_4 \end{bmatrix} \to x_3\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

thus $\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ span $\ker(A)$

Problem 5. *5pts.*

Let $L \subset \mathbb{R}^3$ the line through the origin and $(2,4,6)$ and $x = (4,5,0)$.

i.) (2 points) Find the projection of x onto L.

ii.) (1 point) Find the reflection of x relative to L.

iii.) (2 points) Find a non zero vector $v \in \mathbb{R}^3$ such that v is orthogonal to L and v is also orthogonal to x.

i.) $\text{proj}_L(\vec{x}) = \left(\frac{\vec{x}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\vec{w}$

$\vec{x}\cdot\vec{w} = 8 + 20 + 0 = 28$

$\vec{w}\cdot\vec{w} = 4 + 16 + 36 = 56$

$= \frac{1}{2}(\vec{w}) = \boxed{\begin{bmatrix}1\\2\\3\end{bmatrix}}$

ii.) $\text{ref}_L(\vec{x}) = \vec{x}^{\parallel} - \vec{x}^{\perp} = 2\,\text{proj}_L(\vec{x}) - \vec{x}$

$= 2\begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}4\\5\\0\end{bmatrix} = \begin{bmatrix}2\\4\\6\end{bmatrix} - \begin{bmatrix}4\\5\\0\end{bmatrix}$

$= \boxed{\begin{bmatrix}-2\\1\\6\end{bmatrix}}$

iii.) in order to be orthogonal to both $v \cdot L = 0$ & $v \cdot x = 0$

$\rightarrow$

This can be represented as

$$v_1 2 + v_2 4 + v_3 6 = 0$$

$$v_1 4 + v_2 5 + v_3 0 = 0$$

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 0 \\ 4 & 5 & 0 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -12 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right]$$

$$v_1 - 5v_3 = 0$$

$$v_2 + 4v_3 = 0$$

$$\begin{bmatrix} 5v_3 \\ -4v_3 \\ v_3 \end{bmatrix}$$

any vector that satisfies will be orthogonal.

plug in $v_3 = 1$

$$\boxed{\begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}}$$

is one such vector.