Midterm 2

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Section:

	Tuesday:	Thursday:	
	3A	3B	TA: LI, XIA
	(3C)	3D	TA: ANDERSON, AARON
	3E	3F	TA: JOHNSON, ALEXANDER

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. You may use any available space on the exam for scratch work. If you need more scratch paper, please ask one of the proctors. You must **show your work** to receive credit. Please circle or box your final answers.

Organize your work, in a reasonably neat and coherent way, in the space provided. If you wish for something to not be graded, please strike it out neatly. I will grade only work on the exam paper, unless you clearly indicate your desire for me to grade work on additional pages.

Please do not write below this line.

Problem	Max	Score
1	10	
2	12	
3	12	
4	8	
5	8	
Total	50	

rank(A) + null = 3.

1. (10 pts, 2 pts each) Circle TRUE or FALSE for each statement:

(a) There is a linear transformation $T: \mathbb{R}^6 \to \mathbb{R}^3$ for which $nullity(T) = 2$.

TRUE **FALSE**

(b) Consider a $s \times m$ matrix A and a $t \times m$ matrix B and $C = \begin{bmatrix} A \\ B \end{bmatrix}$, then $\ker(C) \subseteq \ker(A)$.

TRUE **FALSE**

$Ax = 0$.
$Bx = 0$.

(c) Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and $\vec{v}_1, \vec{v}_2 \ldots, \vec{v}_m \in \mathbb{R}^n$ be linear dependent vector, then $T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_m)$ may be linearly independent.

TRUE **FALSE**

(d) Let V and U be two subspaces of $\mathbb{R}^n$, then $\dim(V) + \dim(U) = \dim(V \cap U) + \dim(V + U)$.

TRUE **FALSE**

$AS = SB$. $B = S^{-1}AS$.

(e) Let $T(\vec{x}) = A\vec{x}$ for some matrix $A \in \mathbb{R}^{n \times n}$. If B is similar to A, then B must be a $\mathcal{B}$-matrix of T for some basis $\mathcal{B}$ of $\mathbb{R}^n$.

TRUE **FALSE**

4x3. 3x1 3x4. 4x1 = 3x1.

2. (12 pts)Let V be the subspace of $\mathbb{R}^4$ defined by the following equations

$$\begin{cases} 3x_1 - 9x_2 + 2x_3 - 8x_4 &= 0 \\ x_1 - 3x_2 + 3x_3 + 2x_4 &= 0 \end{cases}$$

(a) (3 pts) Find a linear transformation $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ such that $\ker(T) = V$.

$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = V$

$A = \begin{bmatrix} 3 & -9 & 2 & -8 \\ 1 & -3 & 3 & 2 \\ a & b & c & d \end{bmatrix}$

$= \begin{bmatrix} 3 & -9 & 2 & -8 \\ 1 & -3 & 3 & 2 \\ 4 & -12 & 5 & -6 \end{bmatrix}$

(b) (4 pts)Find a basis for V.

$x_1 - 3x_2 + 3x_3 + 2x_4 = 0$

$x_1 = 3x_2 - 3x_3 - 2x_4$

$3(3x_2 - 3x_3 - 2x_4) - 9x_2 + 2x_3 - 8x_4 = 0$

$9x_2 - 9x_3 - 6x_4 - 9x_2 + 2x_3 - 8x_4 = 0$

$-7x_3 - 14x_4 = 0$

$x_3 = -2x_4$

$x_1 = 3x_2 - 3(-2x_4) - 2x_4$

$= 3x_2 + 6x_4 - 2x_4$

$= 3x_2 + 4x_4$

$\therefore V = \begin{bmatrix} 3x_2 + 4x_4 \\ x_2 \\ -2x_4 \\ x_4 \end{bmatrix}$

$\therefore$ basis for V: $\left(\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right)$

(c) (5 pts) Find a linear transformation S from $\mathbb{R}^3$ to $\mathbb{R}^4$ such that

$$\ker(S) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right) \text{ and } \operatorname{im}(S) = V.$$

$\because \operatorname{im}(S) = V$

$\therefore \operatorname{im}(S) = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

let $A = \begin{bmatrix} 3 & 4 & a \\ 1 & 0 & b \\ 0 & -2 & c \\ 0 & 1 & d \end{bmatrix}$

$A \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = 0$

$\therefore 3 + 4 - a = 0$

$1 + 0 - b = 0$

$0 - 2 - c = 0$

$0 + 1 - d = 0$

$\therefore a = 7$

$b = 1$

$c = -2$

$d = 1$

$\therefore A = \begin{bmatrix} 3 & 4 & 7 \\ 1 & 0 & 1 \\ 0 & -2 & -2 \\ 0 & 1 & 1 \end{bmatrix}$

3. (12 pts) Let A be the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 3 \\ 1 & 3 & 9 \\ 1 & 3 & 5 \end{bmatrix} = [u_1, u_2, u_3] \begin{bmatrix} \|v_1\| & u_1 \cdot v_2 & u_1 \cdot v_3 \\ 0 & \|v_2^\perp\| & u_2 \cdot v_3 \\ 0 & 0 & \|v_3^\perp\| \end{bmatrix}$$

(a) (8 pts) Compute the QR factorization of A.

$u_1 = \frac{1}{\sqrt{1+1+1+1}}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

$\|v_1\| = 2$

$u_1 \cdot v_2 = \frac{1}{2}+\frac{1}{2}+\frac{3}{2}+\frac{3}{2} = 4$

$u_1 \cdot v_3 = \frac{1}{2}-\frac{3}{2}+\frac{9}{2}+\frac{5}{2} = 6$

$v_2^\perp = v_2 - (u_1 \cdot v_2) u_1 = v_2 - 4u_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$\|v_2^\perp\| = \sqrt{1+1+1+1} = 2$

$\therefore u_2 = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$\therefore u_2 \cdot v_3 = -\frac{1}{2}+\frac{3}{2}+\frac{9}{2}+\frac{5}{2} = 8$

$\therefore v_3^\perp = v_3 - ((u_1 \cdot v_3)u_1 + (u_2 \cdot v_3)u_2) = v_3 - (6u_1 + 8u_2) = \begin{bmatrix} 2 \\ 2 \\ 2 \\ -2 \end{bmatrix}$

$\therefore \|v_3^\perp\| = \sqrt{4+4+4+4} = 4$

$u_3 = \frac{1}{4}\begin{bmatrix} 2 \\ 2 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$

ANS $\begin{bmatrix} 2 & 4 & 6 \\ 0 & 2 & 8 \\ 0 & 0 & 4 \end{bmatrix}$

(b) (4 pts) Find the orthogonal projection of $\begin{bmatrix} 3 \\ 5 \\ 5 \\ 1 \end{bmatrix}$ onto im(A).

$\therefore \text{proj} = (u_1 \cdot x)u_1 + (u_2 \cdot x)u_2 + (u_3 \cdot x)u_3$

$\text{proj} = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 0 \end{bmatrix}$

1 3
3 5
9 7
7 1 3
15
= 45 =
5

$\frac{5}{2} - \frac{2}{2} + \frac{5}{2} + \frac{1}{2}$

$= 8 \; 2(u_1)$ ~~80~~ 65 68. $\frac{16}{29\sqrt{5}}$ $\frac{8}{\sqrt{5}}$

$\frac{74}{\sqrt{29}}$ 1 −1 3 3

$\frac{68}{\sqrt{116}} = \frac{68}{2\sqrt{29}}$ 3 5. 15 3 $\frac{4}{\sqrt{116}}$ = $8\sqrt{116}$ 2940 $\frac{16}{\sqrt{20}}$

4. (8 pts) Let $T(\vec{x}) = A\vec{x}$ with $A = \begin{bmatrix} 5 & -2 \\ -1 & 4 \end{bmatrix}$. Consider the basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ with

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

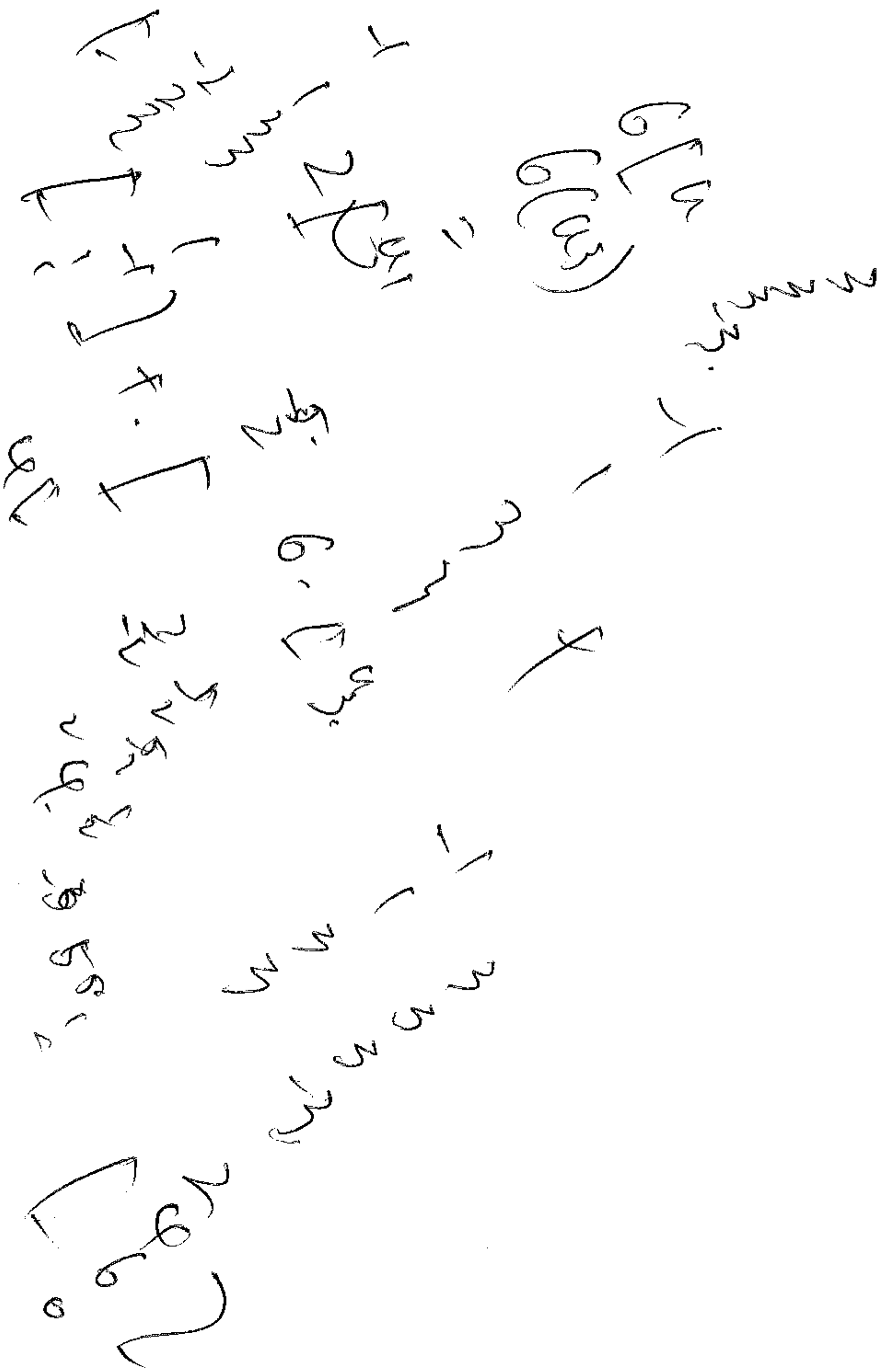
Find the $\mathcal{B}$-matrix B of T.

$$T(v_1) = \begin{bmatrix} 5 & -2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5-2 \\ -1+4 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}_B.$$

$$T(v_2) = \begin{bmatrix} 5 & -2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 10+2 \\ -2-4 \end{bmatrix} = \begin{bmatrix} 12 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}_B.$$

$\therefore$ B- matrix B of T:

$$\begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}.$$



5. (8 pts) Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$ and let V = span(B). Is the vector $\vec{x} = \begin{bmatrix} 7 \\ 6 \\ -1 \\ 4 \end{bmatrix}$ in V ? If so, find $[\vec{x}]_{\mathcal{B}}$.

~~Yes $\vec{x} = \begin{bmatrix} 7 \\ 6 \\ \vdots \\ 4 \end{bmatrix}$ is in V.~~

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ -1 \\ 4 \end{bmatrix}.$$

$c_1 + c_2 = 7.$

$2c_1 = 6. \rightarrow c_1 = 3$

$c_1 - c_2 = -1$

$c_2 = 4. \rightarrow c_2 = 4.$

$\therefore \vec{x} = 3 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 6 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$

$\therefore$ Yes, $\vec{x}$ is in V.

$[\vec{x}]_B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$