

Midterm 1

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Section:

Tuesday:

Thursday:

3A

3B

TA: LI, XIA

3C

3D

TA: ANDERSON, AARON

3E

3F

TA: JOHNSON, ALEXANDER

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators**, books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. You may use any available space on the exam for scratch work. If you need more scratch paper, please ask one of the proctors. You must **show your work** to receive credit. Please circle or box your final answers.

Organize your work, in a reasonably neat and coherent way, in the space provided. If you wish for something to not be graded, please strike it out neatly. I will grade only work on the exam paper, unless you clearly indicate your desire for me to grade work on additional pages.

Please do not write below this line.

Problem	Max	Score
1	15	
2	15	
3	10	
4	10	
Total	50	

Summary of 2D Geometrical Linear Transformations

- Scaling: $kI_2 = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$.
- Projection: $P = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ where $a = u_1^2$, $b = u_1u_2$ and $c = u_2^2 = 1 - u_1^2$ (Unit vector of line $L: \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$)
- Reflection: $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, where $a = 2u_1^2 - 1$ and $b = 2u_1u_2$ (Unit vector of line $L: \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$).
- Rotation combined with a scaling: $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ (Note: scaling factor $r = \sqrt{a^2 + b^2}$, rotation angle $\theta = \tan^{-1}(b/a)$).
- Horizontal shear: $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, vertical shear $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

$$\frac{3-k^2}{3k} \cdot x = \frac{k}{3}$$

$$-7.5 + 13.5 \cdot \frac{3-k^2}{3k} \cdot \frac{k}{3} = \frac{k^2}{3-k^2}$$

$$-7.5 + 4.5 = \frac{k^2}{3-k^2}$$

1. (15 pts) Consider the linear system

$$\begin{cases} 3x_1 + kx_2 = 6 \\ kx_1 + x_2 = -3 \end{cases}$$

$$3x_1 + \sqrt{3}x_2 = 6$$

$$\sqrt{3}x_1 + x_2 = -3$$

$$3x_1 + \sqrt{2}x_2 = -3\sqrt{2}$$

(a) (3 pts) Write the systems in matrix form and write down the corresponding augmented matrix.

$$\begin{bmatrix} 3 & k \\ k & 1 \end{bmatrix} \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3 & k & 6 \\ k & 1 & -3 \end{array} \right]$$

(b) (6 pts) Determine the values of k such that the above linear system is consistent.

~~$\left[\begin{array}{cc|c} 3 & k & 6 \\ k & 1 & -3 \end{array} \right]$~~

$\left[\begin{array}{cc|c} 1 & \frac{k}{3} & 2 \\ k & 1 & -3 \end{array} \right]$ assume $k \neq 0$.

$\rightarrow \left[\begin{array}{cc|c} 1 & \frac{k}{3} & 2 \\ 0 & 1 - \frac{k}{3} & -3 - \frac{2k}{3} \end{array} \right]$

$\rightarrow \left[\begin{array}{cc|c} 1 & \frac{k}{3} & 2 \\ 0 & \frac{3-k}{3} & \frac{-3-k}{3} \end{array} \right]$

$\rightarrow \left[\begin{array}{cc|c} 1 & \frac{k}{3} & 2 \\ 0 & 1 & \frac{-3-k}{3-k} \end{array} \right]$

$\rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 - \frac{k}{3-k} \\ 0 & 1 & \frac{-3-k}{3-k} \end{array} \right]$

$k \neq \pm\sqrt{3}$

(c) (6 pts) If $k = 3$, is the linear system consistent? If it is consistent, please find all the solutions.

Yes it is consistent.

$k = 3$

$$\left[\begin{array}{cc|c} 3 & 3 & 6 \\ 3 & 1 & -3 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 2 & 9 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 - 4.5 \\ 0 & 2 & 9 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 0 & -2.5 \\ 0 & 2 & 9 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 0 & -2.5 \\ 0 & 1 & 4.5 \end{array} \right]$$

$x_1 = -2.5$
 $x_2 = 4.5$

$$\frac{5}{1} - \frac{9}{3} + \frac{7}{1}$$

$$\begin{aligned} v_1 + v_3 &= 2 \\ v_1 + 4v_3 &= 5 \end{aligned} \quad \begin{aligned} v_3 &= 1 \\ v_1 &= 1 \end{aligned}$$

$$v_2 = -2v_2$$

$$v_2 = -2$$

$$v_1 = \frac{5}{1} - \frac{12}{1}$$

$$v_1 + v_3 = 4$$

$$v_1 + 4v_3 = 11$$

$$3v_3 = 7$$

$$v_3 = \frac{7}{3}$$

2. (15 pts) Define the transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ so that

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 1 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 - 2x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 + 3x_2 + 4x_3 \end{bmatrix}$$

(a) (3 pts) Is this transformation linear? If so find its matrix A .

Yes, it is linear.

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 1 \\ 1 & 3 & 4 \end{bmatrix}$$

(b) (7 pts) Is A invertible? If A is invertible, please find A^{-1} .

$$\begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 3 & 4 & 1 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 1 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 3 & 1 & \frac{2}{3} & \frac{1}{3} & 1 \end{bmatrix} \quad \text{Yes, } A \text{ is invertible.}$$

$$\downarrow \quad \downarrow \quad A^{-1} = \begin{bmatrix} \frac{1}{9} & \frac{11}{9} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & 0 \\ \frac{2}{9} & -\frac{5}{9} & \frac{1}{3} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & -1 & 0 \\ 0 & 5 & 3 & 1 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & \frac{1}{9} & \frac{11}{9} & -\frac{1}{3} \\ 0 & 1 & 0 & 1 & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 1 & \frac{2}{9} & -\frac{5}{9} & \frac{1}{3} \end{bmatrix}$$

(c) (5 pts) If there is any, find a vector \vec{v} such that $T(\vec{v}) = \vec{b}$ where

$$\vec{b} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

$$\therefore \cancel{v_1} - 2\cancel{v_2} + \cancel{v_3} = \cancel{v_1} + v_2 + \cancel{v_3}$$

$$-2v_2 = v_2$$

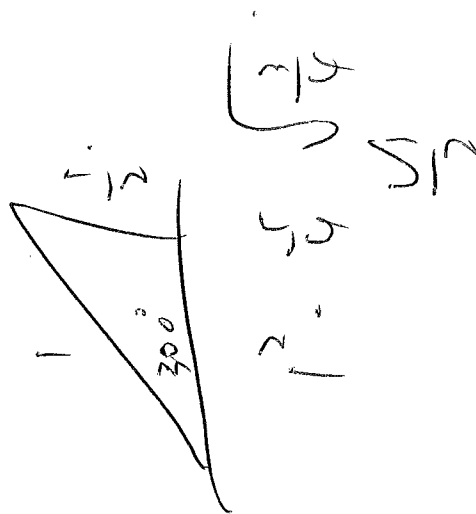
$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 1 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix} \quad \therefore v_1 = v_3 = 1, v_2 = 0$$

$$v_1 - 2v_2 + v_3 = 2$$

$$v_1 + v_2 + v_3 = 2$$

$$v_1 + 3v_2 + 4v_3 = 5$$

$$\therefore \vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$



3. (10 pts) Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a linear transformation for which

$$T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}, \quad T(\vec{e}_1) = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \quad \text{and} \quad T(\vec{e}_2) = \begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix}.$$

Write down an expression for the matrix of the linear transformation from \mathbb{R}^2 to \mathbb{R}^3 described as follows: first rotate all vectors in the plane by 30° counterclockwise, then project onto the line spanned by $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$, then make the resulting vector 3 times longer, and finally apply T . (You do not need to actually multiply the matrices.)

30° counter-clockwise:

$$\begin{bmatrix} \cos(30) & -\sin(30) \\ \sin(30) & \cos(30) \end{bmatrix}$$

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

Assume matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$.

$$\begin{aligned} a_{11} &= 2, & a_{12} &= -4 \\ a_{21} &= -1, & a_{22} &= 2 \\ a_{31} &= 3, & a_{32} &= 1 \end{aligned}$$

$$\therefore A = \begin{bmatrix} 2 & -4 \\ -1 & 2 \\ 3 & 1 \end{bmatrix}$$

~~$$A = \begin{bmatrix} 2 & -4 \\ -1 & 2 \\ 3 & 1 \end{bmatrix}$$~~

projection onto $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$:

$$\frac{1}{9+16} \begin{bmatrix} 9 & -12 \\ -12 & 16 \end{bmatrix}$$

$$= \frac{1}{25} \begin{bmatrix} 9 & -12 \\ -12 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{9}{25} & -\frac{12}{25} \\ -\frac{12}{25} & \frac{16}{25} \end{bmatrix}$$

\therefore the final matrix is:

$$\begin{bmatrix} 2 & -4 \\ -1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 30 \\ 0 \end{bmatrix} \begin{bmatrix} \frac{9}{25} & -\frac{12}{25} \\ -\frac{12}{25} & \frac{16}{25} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

Scaling by 3:

$$\begin{bmatrix} 3 & 0 \end{bmatrix}$$

$$x = 4z, \quad z = 1 \\ y = -1.$$

2x3, 3x1

2x1

4

8

s. -5 - 20.

4. (10 pts) Suppose $T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x + y - 3z \\ 2x + 3y - 5z \end{bmatrix}$. What is $\ker(T)$?

Matrix $A = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 3 & -5 \end{bmatrix}$

$$A\vec{x} = \vec{0}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 2 & 3 & -5 & 0 \end{array} \right]$$

↓

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right]$$

↓

$$\left[\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right]$$

$$x - 4z = 0$$

$$y + z = 0$$

$$\begin{bmatrix} 4z \\ -z \\ z \end{bmatrix} = z \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$$

~~∴ $\ker(T) = \left\{ z \begin{bmatrix} 4z \\ -z \\ z \end{bmatrix} \right\}$~~

~~$\left\{ z \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} \right\}$~~

∴ $\ker(T) = z \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$,

when z is an arbitrary constant.

or that is,

on the line spanned by $\begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$.

~~or~~

~~$\left\{ z \begin{bmatrix} 4z \\ -z \\ z \end{bmatrix} \right\}$~~



