19F-MATH33A-3 Midterm I

KADE ADAMS

TOTAL POINTS

45 / 50

QUESTION 1

1 14 / 15

✓ **- 1 pts** (b) didn't check -3-2k not = 0

✓ **- 0 pts** perfect

QUESTION 2

2 13 / 15

✓ **+ 1 pts** (a) Said "yes" or "linear"

+ 2 pts (a) Correct matrix

✓ **+ 4 pts** (b) Sensible work to compute inverse

✓ **+ 3 pts** (b) Correct inverse

✓ **+ 3 pts** (c) Set up some sensible method (row reduction or multiplying by inverse)

✓ **+ 2 pts** (c) Correct solution vector (give points if based on incorrect inverse, will deduct 1 below)

- 1 pts Used incorrect inverse to find (c)

Points given based on incorrect matrix...

QUESTION 3

3 10 / 10

✓ **- 0 pts** Perfect

QUESTION 4

4 8 / 10

✓ **- 1 pts** The relation between x and z is not right or lack the relation

✓ **- 1 pts** Wrong answer but depends on the wrong matrix/wrong relations

Midterm 1

Last Name: Adams

First Name: Kade

Student ID: 505123178

Signature: Kade Adams

Section:

Tuesday:	Thursday:	
3A	3B	TA: LI, XIA
3C	3D	TA: ANDERSON, AARON
3E	3F	TA: JOHNSON, ALEXANDER

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. You may use any available space on the exam for scratch work. If you need more scratch paper, please ask one of the proctors. You must **show your work** to receive credit. Please circle or box your final answers.
Organize your work, in a reasonably neat and coherent way, in the space provided. If you wish for something to not be graded, please strike it out neatly. I will grade only work on the exam paper, unless you clearly indicate your desire for me to grade work on additional pages.

Please do not write below this line.

Problem	Max	Score
1	15	
2	15	
3	10	
4	10	
Total	50	

Summary of $2D$ Geometrical Linear Transformations

- Scaling: $kI_2 = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$.

- Projection: $P = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ where $a = u_1^2$, $b = u_1u_2$ and $c = u_2^2 = 1 - u_1^2$ (Unit vector of line L: $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$)

- Reflection: $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, where $a = 2u_1^2 - 1$ and $b = 2u_1u_2$ (Unit vector of line L: $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$).

- Rotation combined with a scaling : $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ (Note: scaling factor $r = \sqrt{a^2 + b^2}$, rotation angle $\theta = \tan^{-1}(b/a)$).

- Horizontal shear: $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, vertical shear $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

1. (15 pts) Consider the linear system

$$\begin{cases} 3x_1 + kx_2 = 6 \\ kx_1 + x_2 = -3 \end{cases}$$

(a) (3 pts)Write the systems in matrix form and write down the corresponding augmented matrix.

$$\begin{bmatrix} 3 & k \\ k & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix} \qquad \left[\begin{array}{cc|c} 3 & k & 6 \\ k & 1 & -3 \end{array}\right]$$

(b) (6 pts)Determine the values of k such that the above linear system is consistent.

$$\left[\begin{array}{cc|c} 3 & k & 6 \\ k & 1 & -3 \end{array}\right] \div 3 \rightarrow \left[\begin{array}{cc|c} 1 & k/3 & 2 \\ k & 1 & -3 \end{array}\right] -kr_1 \rightarrow \left[\begin{array}{cc|c} 1 & k/3 & 2 \\ 0 & 1-\frac{k^2}{3} & -3-2k \end{array}\right]$$

$1-\frac{k^2}{3} \neq 0 \quad \frac{k^2}{3} \neq 1 \quad k^2 = 3 \quad k \neq \pm\sqrt{3}$

$\boxed{\text{The system is consistent as long as } k \neq \pm\sqrt{3}}$

(c) (6 pts)If $k = 3$, is the linear system consistent? If it is consistent, please find all the solutions.

$$\left[\begin{array}{cc|c} 3 & 3 & 6 \\ 3 & 1 & -3 \end{array}\right] \div 3 \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 3 & 1 & -3 \end{array}\right] -3r_1 \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & -9 \end{array}\right] \div -2$$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 4.5 \end{array}\right] -r_2 \quad \left[\begin{array}{cc|c} 1 & 0 & -2.5 \\ 0 & 1 & 4.5 \end{array}\right]$$

$\boxed{x_1 = -2.5 \quad x_2 = 4.5 \quad \text{The system is consistent}}$

2. (15 pts)Define the transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ so that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 - 2x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 + 3x_2 + 4x_3 \end{bmatrix}$$

(a) (3 pts)Is this transormation linear? If so find its matrix A.

Transformation is linear

$$x_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

$-1-(3\cdot -\tfrac{1}{3}) = -1-(-1) = 0$

(b) (7 pts) Is A invertible? If A is invertible, please find A^{-1}.

A is invertible if rref(A) $= I_n$

$[A|I]$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \\ -r_1 \\ -r_1 \end{matrix} \to \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 3 & 0 & -1 & 1 & 0 \\ 0 & 3 & 3 & -1 & 0 & 1 \end{array}\right] \div 3 \to \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\tfrac{1}{3} & \tfrac{1}{3} & 0 \\ 0 & 3 & 3 & -1 & 0 & 1 \end{array}\right] \begin{matrix} +2r_2 \\ \\ -3r_2 \end{matrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \tfrac{1}{3} & \tfrac{2}{3} & 0 \\ 0 & 1 & 0 & -\tfrac{1}{3} & \tfrac{1}{3} & 0 \\ 0 & 0 & 3 & 0 & -1 & 1 \end{array}\right] \div 3 \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \tfrac{1}{3} & \tfrac{2}{3} & 0 \\ 0 & 1 & 0 & -\tfrac{1}{3} & \tfrac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & -\tfrac{1}{3} & \tfrac{1}{3} \end{array}\right] -r_3 \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \tfrac{1}{3} & 1 & -\tfrac{1}{3} \\ 0 & 1 & 0 & -\tfrac{1}{3} & \tfrac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & -\tfrac{1}{3} & \tfrac{1}{3} \end{array}\right]$$

rref(A) $= I_n$ so $A^{-1} = \begin{bmatrix} \tfrac{1}{3} & 1 & -\tfrac{1}{3} \\ -\tfrac{1}{3} & \tfrac{1}{3} & 0 \\ 0 & -\tfrac{1}{3} & \tfrac{1}{3} \end{bmatrix}$

(c) (5 pts) If there is any, find a vector $\vec{v}$ such that $T(\vec{v}) = \vec{b}$ where

$$\vec{b} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 4 & 5 \end{array}\right] \begin{matrix} \\ -r_1 \\ -r_1 \end{matrix} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & 3 & 0 & 0 \\ 0 & 3 & 3 & 3 \end{array}\right] \div 3 \to \left[\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 3 & 3 \end{array}\right] \begin{matrix} +2r_2 \\ \\ -3r_2 \end{matrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 3 \end{array}\right] \div 3 \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] -r_3 \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\boxed{\vec{V} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}$$

3. (10 pts) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation for which

$$T(\vec{e}_1) = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \quad \text{and} \quad T(\vec{e}_2) = \begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix}.$$

Write down an expression for the matrix of the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ described as follows: first rotate all vectors in the plane by 30° counterclockwise, then project onto the line spanned by $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$, then make the resulting vector 3 times longer, and finally apply T. (You do not need to actually multiply the matrices.)

Projection $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$

$\text{scale}(\text{proj}_L(\text{rot}\,\vec{x})) = k(P(R\vec{x})) = (kPR)\vec{x}$

$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix}$

$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 9/25 & -12/25 \\ -12/25 & 16/25 \end{bmatrix}\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

rotation $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$T = \begin{bmatrix} 2 & -4 \\ -1 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\cos(30°) = \frac{\sqrt{3}}{2}$

$\sin(30°) = \frac{1}{2}$

$$\boxed{\begin{bmatrix} 2 & -4 \\ -1 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 9/25 & -12/25 \\ -12/25 & 16/25 \end{bmatrix}\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

4. (10 pts) Suppose $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y-3z \\ 2x+3y-5z \end{bmatrix}$. What is $\ker(T)$?

$$x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 1 \\ 3 \end{bmatrix} + z\begin{bmatrix} -3 \\ -5 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 3 & -5 \end{bmatrix} \qquad T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = 0$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 2 & 3 & -5 & 0 \end{array}\right]_{-2r_1} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]^{-r_2} \left[\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$x - 4y = 0 \qquad y = \frac{x}{4} \qquad \begin{bmatrix} x \\ \frac{x}{4} \\ -y \end{bmatrix}$$

$$x + z = 0 \qquad z = -x$$

$$\boxed{\ker(T) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1/4 \\ -1 \end{bmatrix}\right)}$$