33A Killip Midterm 2 November 19th

First Name: ____________________ ID#____________________

Last Name: ____________________

Section: ____________________ $= \begin{cases} a & \text{Tuesday with Laackman} \\ b & \text{Thursday with Laackman} \\ c & \text{Tuesday with Sella} \\ d & \text{Thursday with Sella} \\ e & \text{Tuesday with Bellis} \\ f & \text{Thursday with Bellis} \end{cases}$

Rules.

- This is a 50 minute exam.
- There are **FIVE** problems; ten points per problem.
- There are extra pages each problem. You may also use the backs of pages.
- No calculators, computers, notes, books, crib-sheets,...
- Out of consideration for your class-mates, no chewing, humming, pen-twirling, snoring,... Try to sit still.
- Turn off your cell-phone, pager,...

1	2	3	4	5	Σ
8	7	9	10	7	41

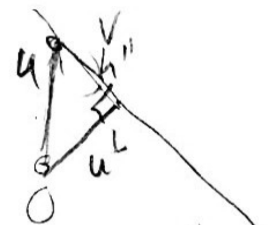
(1) (a) True / False: Suppose A is invertible and has QR decomposition $A = QR$, then the columns of Q^T are orthonormal.

(b) $\det\left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}\right) = \underline{\quad 0 \quad}$

(c) Let $\vec{u}$ and $\vec{v}$ be non-zero vectors in $\mathbb{R}^n$. For which number t is

$$\vec{u} + t\vec{v}$$

closest to the origin?



$$t = \frac{-(u \cdot v)}{v \cdot v}$$

$$u - u^{\parallel} = u - \frac{u \cdot v}{\|v\|^2} v$$

t

(d) The volume of the parallelepiped generated by $\begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 7 \\ 2 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is $\underline{\quad 42 \quad}$

$$\det \begin{bmatrix} 7 & 7 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} = 7 \times 2 \times 3 = 42$$

(e) For $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$, find the vector in $\text{im}(A)$ that is closest to $\vec{0}$.

$\vec{0}$

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(2) Let $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \\ 0 & -4/5 & 3/5 \end{bmatrix}$ and suppose we have two bases $\mathcal{B}$ and $\mathcal{E}$ for $\mathbb{R}^3$ that are related by

$$\sqrt{\frac{9}{25}+\frac{16}{25}} = \frac{25}{25} = 1$$

$$[\vec{x}]_{\mathcal{B}} = C[\vec{x}]_{\mathcal{E}}$$

(a) True / False: C an orthogonal matrix.

(b) True / False: C^{-1} and C^T commute.

(c) If $[\vec{x}]_{\mathcal{B}} \cdot [\vec{y}]_{\mathcal{B}} = 7$ for some specific vectors $\vec{x}$ and $\vec{y}$, what is $[\vec{x}]_{\mathcal{E}} \cdot [\vec{y}]_{\mathcal{E}}$?

(d) Let T be a linear transformation. Compute $[T]_{\mathcal{B}}$ given that

$$[T]_{\mathcal{E}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

c. $x_B \cdot y_B = 7$

$\cancel{C[x]_E \cdot C[y]_E = 7}$

$\cancel{[x_E][y]_E = 7}$

$-\frac{16}{5} + \frac{9}{5}$ $\quad -\frac{6}{9} + \frac{12}{5}$ $\quad -\frac{24}{25} + \frac{12}{5}$ $\quad \frac{60}{25}$

$-\frac{2}{3} + \frac{12}{5}$ $\quad \frac{-10 \quad 36}{15}$ $\quad -\frac{30}{9\ 45}$ $\quad -\frac{16}{5}$ $\quad -\frac{6}{9} - \frac{16}{5}$ $\quad -\frac{2}{3} - \frac{16}{5}$ $\quad \frac{-10-48}{15}$

d. $[T]_B = C[T]_E C^{-1}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3/5 & 4/5 \\ 0 & -4/5 & 3/5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} C^{-1}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6/5 & 4 \\ 0 & -8/5 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -5/9 & -4/5 \\ 0 & 5/3 & 3/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{58}{5} & \frac{36}{25} \\ 0 & -\frac{24}{15} & -\frac{7}{5} \end{bmatrix}$

$$C^{-1} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3/5 & 4/5 & 0 & 1 & 0 \\ 0 & -4/5 & 3/5 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 4 & 0 & 5 & 0 \\ 0 & -4 & 3 & 0 & 0 & 5 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4/3 & 0 & 5/3 & 0 \\ 0 & -1 & 3/4 & 0 & 0 & 5/4 \end{array}\right]$$

extra paper

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 12 & 16 & 0 & 20 & 0 \\ 0 & -12 & 9 & 0 & 0 & 15 \end{array}\right]$$

$$\frac{4}{3}\ \frac{3}{4} = 6 \quad \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 12 & 16 & 0 & 20 & 0 \\ 0 & 0 & 25 & 0 & 20 & 15 \end{array}\right]$$

$$\frac{20}{12} = \frac{5}{3}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4/3 & 0 & 5/3 & 0 \\ 0 & 0 & 1 & 0 & 5/3 & 3/5 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -5/9 & -\frac{4}{5} \\ 0 & 1 & 1 & 0 & 5/3 & 3/5 \end{array}\right]$$

$$-4/3\left(5/3 \qquad 3/5\right)$$

$$-\frac{20}{9} \qquad -\frac{12}{15}$$

(3) Answer the questions below about the following matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

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(a) Find a basis for im(A)

(b) Find the vector $\vec{v}$ in im(A) that is closest to $\begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$.

$a.\ \text{im}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \cancel{\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}}, \cancel{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}} \right)$

$\vec{v}_4 = \vec{v}_1 + \vec{v}_3$

$\cancel{v_3 = v_1 - 2v_2}$

$\vec{u}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$\vec{v}_2^{\perp} = \vec{v}_2 - v_2^{\parallel} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{4}(2)\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$

$\vec{u}_2 = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$

$\vec{v}_3^{\perp} = \vec{v}_3 - \vec{v}_3^{\parallel} = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{4}(-1)\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/4 \\ -3/4 \\ -3/4 \\ 5/4 \end{bmatrix}$

$\|v_3^{\perp}\| = \sqrt{\frac{1}{16} + \frac{9}{16} + \frac{9}{16} + \frac{25}{16}}$

$1 + 9 + 9 + 25$

$\sqrt{\frac{44}{16}}$

$\frac{11}{4}$

$\vec{u}_3 = \frac{1}{\sqrt{\frac{11}{4}}}\begin{bmatrix} 1/4 \\ -3/4 \\ -3/4 \\ 5/4 \end{bmatrix}$

$\text{im}(A) = \text{span}\left(\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}, \frac{1}{\sqrt{\frac{11}{4}}}\begin{bmatrix} 1/4 \\ -3/4 \\ -3/4 \\ 5/4 \end{bmatrix} \right)$

$\vec{x} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$

b.

$\vec{v} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + (u_2 \cdot x)u_2 + (u_3 \cdot x)u_3$

$= 2\left(\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right) + 0 + \frac{4}{11}\left[\frac{2}{4} - \frac{6}{4}\right] \begin{bmatrix} 1/4 \\ -3/4 \\ -3/4 \\ 5/4 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{-4 \times 4}{11 \times 4}$

$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + -\frac{4}{11}\begin{bmatrix} 1/4 \\ -3/4 \\ -3/4 \\ 5/4 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{11} \\ \\ \end{bmatrix}$

a. $\operatorname{im}(A) = \operatorname{span}\left(\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}\right)$ ✓

b. $\vec{v} = (u_1 \cdot x)u_1 + (u_2 \cdot x)u_2$

$= 2\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

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(4) (a) Finish the following *definition*:

We say that the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are *orthonormal* if ...

each vector is orthogonal to the others and has a length of one.

(b) **True** / **False** : Orthonormal vectors are linearly independent.

(c) Compute $\det\left(\begin{bmatrix} 1 & 1 & 5 & 6 \\ 0 & 2 & 3 & 7 \\ 0 & 4 & 5 & 8 \\ 2 & 2 & 10 & 9 \end{bmatrix}\right)$. $\to \det \begin{bmatrix} 1 & 1 & 5 & 6 \\ 0 & 2 & 3 & 7 \\ 0 & 0 & -1 & -6 \\ 0 & 0 & 0 & -3 \end{bmatrix} = 1 \times 2 \times -1 \times -3 = 6$

(5) (a) Perform a QR decomposition of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 3 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

$A = QR$

$\vec{u}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$ ✓

$v_2^{\perp} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$\vec{u}_2 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$ ✓ $= \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$

$-(u_2 \cdot v_3)\, u_2$

$v_3^{\perp} = \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} - 2\begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$

$\vec{u}_3 = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$ ✗

$$Q = \begin{bmatrix} 1/2 & 1/2 & -1/\sqrt{6} \\ 1/2 & -1/2 & 2/\sqrt{6} \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & -1/\sqrt{6} \end{bmatrix}$$

$u_1 \cdot v_2$ $\quad u_1 \cdot v_3^{\perp}$ $\quad u_2 \cdot v_3$

$$R = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & \sqrt{6} \end{bmatrix}$$